Estimation of SVJD Models with Bayesian Methods and Power-Variation Estimators

Milan Fičura Jiří Witzany

1.1 Introduction

Stochastic-Volatility Jump-Diffusion (SVJD) models have become an established tool for simulating the paths of the future asset price evolution which can subsequently be used for tasks such as Value at Risk calculation, option pricing, volatility forecasting or quantitative trading. As the SVJD model estimation includes the estimation of the evolution of several series of latent state variables (i.e. the unobservable stochastic variances, jump occurrences, jump sizes, etc.), it is most commonly performed with Bayesian methods such as Markov-Chain Monte-Carlo (MCMC) and Particle Filters, with the MCMC being used for the in-sample estimation [Eraker, Johannes and Polson, 2003; Witzany, 2013], while the Particle Filters can be used for the sequential estimation of the latent state variables for the purpose of predictions [Fulop, Li and Yu, 2015].

Independently to the advances in the estimation methods for SVJD models, an alternative volatility and jump modelling framework emerged in the recent years, utilizing high-frequency returns and the asymptotic theory of power variations [Barndorff-Nielsen and Shephard, 2004; Andersen, Bollerslev and Diebold, 2007]. The approach consists of an estimation of the volatility and the jump component of the asset returns with power-variation estimators that converge to these quantities when the return sampling frequency converges to infinity. By utilizing the high-frequency returns it is then possible to get very accurate estimates of the evolution of the asset price volatility and jumps, which can then be modelled with standard time series models as if they were observable.

Nevertheless, treating the inherently noisy power-variation estimators as if they were corresponding to the latent stochastic variances and jumps may be sub-optimal in certain cases. Additionally, in applications where the estimation of the future asset price distribution is required (i.e. option pricing, VaR estimation, etc.), it may still be useful to estimate a full SVJD model with Bayesian methods, that can easily be used to perform simulations

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of the future asset price evolution, while taking into account the uncertainties in the parameter estimation and the uncertainty about the current values of the latent states.

Due to the reasons mentioned above, we present an approach of how to utilize the information from the high-frequency power-variation estimators as an additional source of information in the Bayesian estimation of SVJD models. The fit of these models is expected to better correspond to the past evolution of the stochastic variances and jumps, as it uses significantly more accurate estimates for their estimation, while at the same time keeping all of the benefits of the SVJD model framework.

Two models are proposed, the SVJD-RV model [Takahashi, Omori and Watanabe, 2009], utilizing the realized variance estimator as an additional source of information for the estimation of the stochastic variances, and a SVJD-RV-Z model [Fičura and Witzany, 2015], utilizing additionally the Z-Statistics estimator as an additional source of information for the estimation of jumps. Both of the models are compared with a standard SVJD model (using only the information about the daily returns), on simulated as well as empirical foreign exchange rate time series.

The purpose of the simulation study is to asses, whether the extended models (SVJD-RV and SVJD-RV-Z) provide more accurate estimates of the underlying stochastic variances and jumps than the basic SVJD model and non-parametric approaches. MCMC algorithm is used to estimate the models on simulated time series with different jump magnitudes and the fit to the simulated variances and jumps is then evaluated.

In the empirical study performed on foreign exchange rates (EUR/USD, GBP/USD, USD/CHF and USD/JPY), the focus is placed on the evaluation of the out-sample predictive power of the models with regards to the future quadratic variations (approximated with realized variances). MCMC algorithm is used to estimate the model parameters and latent state variables on the in-sample period, while Particle Filters are used to sequentially estimate the latent state variable evolution in the out-sample period. Simulations are then used to calculate forecasts of the future quadratic variation at different horizons at each time point, with the forecast accuracy of the models evaluated with the R-Squared criterion.

The rest of the chapter is organized as follows. In the next subsection, the realized variance estimator and the Z-Estimator of jumps are presented. In the following two subsections, the SVJD, SVJD-RV and SVJD-RV-Z models are explained, and the MCMC and Particle Filter algorithms used for their estimation are described. The following two subsections contain the simulation study of the model in-sample fit and the empirical study of their out-sample predictive power. The last subsection provides a conclusion of the main results of the performed studies.

1.2 Power-Variation Estimators

Power-Variation estimators represent a non-parametric, model-free approach to estimate the volatility and jump components of financial time series. In the last two decades they have proved to be extremely useful in a wide variety of applications. In our current study

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the estimators are used as additional sources of information (in addition to the daily returns) for the estimation of Stochastic-Volatility Jump-Diffusion (SVJD) models.

Let us assume that the logarithmic price of an asset follows a generally defined Stochastic-Volatility Jump-Diffusion process expressed with the following differential equation:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + j(t)dq(t) \quad (1)$$

where $p(t)$ is the logarithm of the asset price, $\mu(t)$ is the instantaneous drift rate, $\sigma(t)$ is the instantaneous volatility, $W(t)$ is a Wiener process, $j(t)$ is a process determining the size of the jumps and $q(t)$ is a counting process whose differential determines the times of the jump occurrences.

The total variability of the price process over a period of time between $t-1$ and t can be expressed with its *quadratic variation*, defined as follows:

$$QV(t) = \int_{t-1}^{t} \sigma^2(s)ds + \sum_{t-1 \le s < t} \kappa^2(s), \quad (2)$$

where the first term on the right side, representing the continuous component of the price variability, is called *integrated variance*, while the second term, representing the discontinuous component of the price variability (i.e. the effect of the jumps) is called *jump variance*. We can thus rewrite the equation as follows:

$$QV(t) = IV(t) + JV(t) \quad (3)$$

where $IV(t)$ denotes the integrated variance and $JV(t)$ the jump variance.

As all of the quantities, $QV(t)$, $IV(t)$ and $JV(t)$, are directly unobservable from the asset price evolution, they have to be estimated. In the case when only the daily price returns are available, this is a challenging task and Bayesian methods have to be used in order to estimate the posterior distribution of the latent state time series of the stochastic variances, jump occurrences and jump sizes. In the case when the intraday, high-frequency data are available, however, it is possible to utilize the asymptotic theory of power variations in order to construct estimators that converge to the target quantities, i.e. either to $QV(t)$, $IV(t)$ or $JV(t)$, when the return sampling frequency increases. If we could increase the sampling frequency to infinity, these estimators would provide perfect estimates of the underlying quantities. In practical settings, infinitely high sampling frequency is unachievable, high-frequency returns (tick, 1-minute, 5-minute, etc.) are thus commonly being used causing the estimators to be plagued by certain levels of noise, and the estimators may sometimes even be biased due to microstructure noise effects (finiteness of the price grid, bid-ask bounce, etc.). In spite of this, power-variation estimators proved to be sufficiently accurate in practice and are currently being commonly used in a wide variety of tasks, especially in the area of volatility forecasting [Andersen, Bollerslev and Diebold, 2007].

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In the presented study, the *realized variance* estimator is used to estimate the quadratic variation of the underlying price process. Realized variance, $RV(t, \Delta)$, proposed by Andersen and Bollerslev [1998], is defined, for a given frequency, as the sum of squared returns on some higher frequency, and it should converge to the quadratic variation of the price process (on the lower frequency), when the sampling frequency of the higher-frequency returns increases.

Specifically, denoting Δ as some intraday time interval and $r(t, \Delta)$ as the logarithmic return between $t - \Delta$ and t, we can define the realized variance as follows:

$$\mathrm{RV}(\mathrm{t}, \Delta) = \sum_{j=1}^{1/\Delta} \mathrm{r}^2(\mathrm{t} - 1 + \mathrm{j}\Delta, \Delta), \tag{4}$$

and it holds that $\mathrm{RV}(\mathrm{t}, \Delta) \rightarrow \mathrm{QV}(\mathrm{t})$ as $\Delta \rightarrow 0$.

The estimation of the jump component of the time series is less straightforward and it is commonly performed by using the difference between an estimate of $QV(t)$ and $IV(t)$. As already mentioned, the integrated quarticity can be estimated with the realized variance. To estimate the integrated variance, a *realized bipower variation* is commonly used (Barndorff-Nielsen and Shephard, 2004). It can be defined as follows:

$$\mathrm{BV}(\mathrm{t}, \Delta) = \frac{\pi}{2} \sum_{j=2}^{1/\Delta} |\mathrm{r}(\mathrm{t} - 1 + \mathrm{j}\Delta, \Delta)||\mathrm{r}(\mathrm{t} - 1 + (\mathrm{j} - 1)\Delta, \Delta)|, \tag{5}$$

and it holds that $\mathrm{BV}(\mathrm{t}, \Delta) \rightarrow \mathrm{IV}(\mathrm{t})$ as $\Delta \rightarrow 0$.

The contribution of the jump component to the time series variability (i.e. the jump variance) can thus be roughly estimated as:

$$\mathrm{RJV}(\mathrm{t}, \Delta) = \mathrm{RV}(\mathrm{t}, \Delta) - \mathrm{BV}(\mathrm{t}, \Delta), \tag{6}$$

where $\mathrm{RJV}(\mathrm{t}, \Delta)$ is the realized jump variance and $\mathrm{RJV}(\mathrm{t}, \Delta) \rightarrow \sum_{t-1 \leq s < t} \kappa^2(s)$ as $\Delta \rightarrow 0$. Nevertheless, as long as we are not able to sample the asset returns at an infinitely high sampling frequency, the estimates of $\mathrm{RV}(\mathrm{t}, \Delta)$ and $\mathrm{BV}(\mathrm{t}, \Delta)$ are inherently plagued by some noise, causing the value of $\mathrm{RJV}(\mathrm{t}, \Delta)$ to differ from zero in almost all of the days in the time series (reaching even negative values, when the noise causes the bipower variation to be greater than the realized variance).

As we would like the to pick only the statistically significant jumps occurring in the time series, it has been proposed to normalize the estimator and use only the values when it is statistically significantly larger than zero, indicating a presence of a jump (Barndorff-Nielsen and Shephard, 2004). In order to determine whether a given value $\mathrm{RJV}(\mathrm{t}, \Delta)$ is significantly different from zero or not, it is necessary to model the volatility of the estimator which can be done by using the *integrated quarticity*:

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$$\mathrm{IQ}(t) = \int_{t-1}^{t} \sigma^4(s) ds \quad (7)$$

which can consistently be estimated, even in the presence of jumps, with the *realize tri-power quarticity* [Andersen, Bollerslev and Diebold, 2007], defined as follows:

$$\mathrm{TQ}(\mathrm{t},\Delta) = \frac{\pi^{3/2}}{4\Delta}\Gamma\left(\frac{7}{6}\right)^{-3} \sum_{j=3}^{1/\Delta} |\mathrm{r}(\mathrm{t}-1+\mathrm{j}\Delta,\Delta)|^{4/3} |\mathrm{r}(\mathrm{t}-1+(\mathrm{j}-1)\Delta,\Delta)|^{4/3} |\mathrm{r}(\mathrm{t}-1+(\mathrm{j}-2)\Delta,\Delta)|^{4/3}, \quad (8)$$

as it holds that $\mathrm{TQ}(\mathrm{t},\Delta) \to \mathrm{IQ}(t)$ when $\Delta \to 0$.

Using the $\mathrm{RV}(\mathrm{t},\Delta)$, $\mathrm{BV}(\mathrm{t},\Delta)$ and $\mathrm{TQ}(\mathrm{t},\Delta)$, it is possible to define the so called Z-Estimator of jumps, $\mathrm{Z}(\mathrm{t},\Delta)$, developed by Barndorff-Nielsen and Shephard [2004] and Andersen, Bollerslev and Diebold [2007]. The estimator uses appropriately normalized relative differences between $\mathrm{RV}(\mathrm{t},\Delta)$ and $\mathrm{BV}(\mathrm{t},\Delta)$, that should asymptotically follow a standard normal distribution as long as the underlying price process does not contain jumps. The estimator is defined as:

$$Z(\mathrm{t},\Delta) = \frac{[RV(t,\Delta) - B\mathrm{V}(\mathrm{t},\Delta)]RV(t,\Delta)^{-1}}{\sqrt{[(\pi/2)^2 + \pi - 5]\max\{1, TV(\mathrm{t},\Delta)B\mathrm{V}(\mathrm{t},\Delta)^{-2}\}\Delta}} \quad (9)$$

As in the case of no jumps in the time series, the asymptotic distribution of $Z(\mathrm{t},\Delta)$ is the standard normal, it is possible to identify jumps in the time series based on the days when the values of $Z(\mathrm{t},\Delta)$ surpass a certain sufficiently high quantile α of the standard normal distribution. The jump variance $\mathrm{EJV}(\mathrm{t},\Delta)$ can thus be estimated as:

$$\mathrm{EJV}(\mathrm{t},\Delta) = I\{Z(\mathrm{t},\Delta) > \Phi(\alpha)^{-1}\}[\mathrm{RV}(\mathrm{t},\Delta) - \mathrm{BV}(\mathrm{t},\Delta)] \quad (10)$$

where $\mathrm{EJV}(\mathrm{t},\Delta)$ is the jump variance estimator (whose non-zero values correspond to jump occurrences), $I\{.\}$ is the indicator function and $\Phi(\alpha)^{-1}$ is the quantile function of the standard normal distribution, with α being the significance level used for the jump estimation.

As we want the sum of the jump estimator and the integrated variance estimator to be equal to the realized variance, we need to re-estimate the integrated variance as:

$$\mathrm{EIV}(\mathrm{t},\Delta) = \mathrm{RV}(\mathrm{t},\Delta) - I\{Z(\mathrm{t},\Delta) > \Phi(\alpha)^{-1}\}[\mathrm{RV}(\mathrm{t},\Delta) - \mathrm{BV}(\mathrm{t},\Delta)] \quad (11)$$

In the following parts of the study, the values of $\mathrm{RV}(\mathrm{t},\Delta)$, $\mathrm{EIV}(\mathrm{t},\Delta)$ and $\mathrm{EJV}(\mathrm{t},\Delta)$ are used as additional sources of information for the estimation of the latent state variables (stochastic variances, jump occurrences and jump sizes) in the extended versions of the basic SVJD model (namely in the SVJD-RV model and the SVJD-RV-Z model).

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1.3 SVJD, SVJD-RV and SVJD-RV-Z Models

In this section, the Stochastic-Volatility Jump-Diffusion (SVJD) model with self-exciting jumps is explained as well as two extended versions of the model, utilizing either the realized variance (SVJD-RV) or the realized variance together with the Z-Estimator of jumps (SVJD-RV-Z) as additional sources of information for the estimation of the latent state time series of the stochastic variances, jump occurrences and jump sizes. Due to the latent state time series in the models, they need to be estimated with Bayesian methods such as the MCMC algorithm and Particle Filters which are explained in detail in the next section. The basic SVJD model with self-exciting jumps consists of 3 equations, one determining the behaviour of the logarithmic returns, one determining the behaviour of the stochastic variances and one determining the intensity of the jump occurrences. The stochastic variances are, in the presented approach, expected to follow the log-variance model, while the jumps follow a self-exciting Hawkes process [Fulop, Li and Yu, 2015]. The SVJD-RV model adds a fourth equation into the model, determining the relationship between the realized variances and the stochastic variances [Takahashi, Omori and Watanabe, 2009], while the SVJD-RV-Z model adds an additional fifth equation, determining the relationship between the Z-Statistics and the jump occurrences [Ficura and Witzany, 2015]. In all of the cases, the models are formulated in discrete time, although the original SVJD model can be formulated in the continuous time as well.

The first equation, governing the evolution of the log-returns, is defined as follows:

$$r(t) = \mu + \sigma(t)\varepsilon(t) + J(t)Q(t) \tag{12}$$

where $r(t)$ is the daily logarithmic return, defined as $r(t) = p(t) - p(t-1)$, with $p(t)$ being the logarithm of the closing price at day t. Parameter μ determines the unconditional mean daily return, $\sigma(t)$ is the conditional volatility, $\varepsilon(t) \sim N(0,1)$ is a standard normal random variable, $J(t) \sim N(\mu_J, \sigma_J)$ is a normally distributed random variable determining the jump sizes and $Q(t) \sim \mathrm{Bern}[\lambda(t)]$ is a variable determining the times of jump occurrences, following a Bernoulli process with intensity $\lambda(t)$.

The second equation describes the evolution of the daily conditional variances. In the presented approach, these are assumed to follow a log-variance model, in which the logarithm of the daily conditional variance $h(t) = \ln[\sigma^2(t)]$, follows an AR(1) process. The conditional variance equation is thus as follows:

$$h(t) = \alpha + \beta h(t-1) + \gamma \varepsilon_V(t) \tag{13}$$

where $h(t) = \ln[\sigma^2(t)]$ is the logarithm of the conditional variance of the daily returns, α is the constant, which is linked to the unconditional variance via $\alpha = (1-\beta)\theta$, where θ denotes the long-term unconditional log-variance, β is the autoregressive coefficient of the AR(1) model, γ is the volatility of the log-variance and $\varepsilon_V(t) \sim N(0,1)$ is a series of standard normal random variables for which we assume to be uncorrelated with $\varepsilon(t)$

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as we work with foreign exchange rate time series for which no long-term correlation between the returns and volatility seems to be present.
The third equation models the jump intensity as implied by the self-exciting Hawkes process that is used to model the jump behaviour. The Hawkes process allows us to model the jump clustering effect by assuming that the jumps self-excite in the sense that an occurrence of a jump temporarily increases the probability of further jumps. The discrete version of the Hawkes process jump intensity $\lambda(t)$ is:

$$\lambda(t) = \alpha_J + \beta_J \lambda(t-1) + \gamma_J Q(t-1) \tag{14}$$

where $\lambda(t)$ is the jump intensity at day t, α_J is the constant, which is linked to the long-term jump intensity θ_J via $\alpha_J = (1 - \beta_J - \gamma_J)\theta_J$, parameter β_J is the rate of the exponential decay of the jump intensity, and γ_J is the increase of jump intensity in the day following a jump occurrence. As we can see, in the day immediately following a jump occurrence, the jump intensity increases by γ_J, decaying in the subsequent days gradually back to its long-term level θ_J at an exponential rate β_J.
Equations 12, 13 and 14 fully characterize the standard SVJD model with self-exciting jumps. The following two equations correspond to the SVJD-RV model [Takahashi, Omori and Watanabe, 2009] and the SVJD-RV-Z model [Ficura and Witzany, 2015].
The fourth equation establishes a link between the realized variance, adjusted for the occurred jumps, and the stochastic variance of the price process. It is defined as follows:

$$\log[RV(t) - J^2(t)Q(t)] = h(t) + \sigma_{RV}\varepsilon_{\mathrm{RV}}(t) \tag{15}$$

The equation 16 can be derived from the definitions of the quadratic variation and the realized variance, and it tells us that the logarithm of the realized variance, adjusted for the estimated jump variance (i.e. the squared jump component of the SVJD model), provides an unbiased estimate of the underlying stochastic log-variance $h(t)$. The realized variance estimator is, however, assumed to be plagued by some estimation noise, given by $\varepsilon_{RV}(t) \sim N(0,1)$, multiplied with the standard deviation of the noise σ_{RV}.
Although in practical applications may the realized variance be a biased estimator of the quadratic variation, due to the microstructure noise effects, present at the ultra-high frequencies, the presented approach assumes the estimator to be unbiased (which is approximately true if lower frequencies, such as the 15-minute one, are used for the RV estimation). Nevertheless, a bias of the RV estimator could easily be incorporated into the model if needed, by adding a constant (i.e. additional parameter) to the right-hand side of the equation. The approach would then be robust to the microstructure noise related bias and may thus be applied even to the higher-frequency returns.
The fifth equation of the model establishes a link between the values of the Z-Statistics and the jumps estimated by the SVJD-RV-Z model. A methodological problem had to be solved in this case, as the Z-Statistics, working on the intraday frequencies, has the tendency of indicating certain jump occurrences on almost every day in the time series, due

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to the small jumps occurring at the very high-frequencies. As the SVJD model works on the daily frequency, we are not interested in the small intraday jumps so much, but rather want to estimate the large jumps, with a significant impact on the distribution of the daily returns. In order to utilize the Z-Statistic for a more accurate estimation of the large jumps, influencing the daily return distribution, we utilize the fact that higher jumps tend to increase the Z-Statistics more than the small jumps. Different mean values of the Z-Statistic can thus be expected in the days in which there are either no jumps or only small, intraday jumps, then on the days when the large price jumps occur. This leads us to the following relationship between the $Z(\mathrm{t})$ and the $Q(\mathrm{t})$:

$$Z(t) = \mu_Z + \xi_Z Q(t) + \sigma_Z \varepsilon_Z(t) \tag{16}$$

where μ_Z corresponds to the mean value of $Z(t)$ on the days with either no jumps, or only small jumps, that are not visible on the daily frequency, so the daily-frequency jump estimate $Q(t)$ equals to zero, while ξ_Z measures the increase in the mean value of $Z(\mathrm{t})$ in the days when the large jumps occur and $Q(t)$ is thus equal to one. Additionally, σ_Z corresponds to the volatility of the $Z(t)$, which may be different from 1 due to the effect of the jumps, while $\varepsilon_{RV}(t) \sim N(0,1)$ is a Gaussian white noise.

Equations 12-15 characterize the SVJD-RV model, while the equations 12-16 correspond to the SVJD-RV-Z model.

1.4 Bayesian Estimation of SVJD Models

In order to estimate the parameters of the proposed SVJD models and the series of their latent state variables (stochastic variances, jump occurrences and jump sizes), Bayesian estimation methods are utilized. Specifically, a MCMC algorithm is used to estimate the parameters of the models and the evolution of the latent state variables in the in-sample period [Witzany, 2013)], while a Sequential Importance Resampling (SIR) particle filter is used to sequentially estimate the evolution of the latent state variables in the out-sample period, with simulations being used to construct predictions.

MCMC is a Bayesian estimation method that enables us to sample from the high-dimensional joint posterior density of the model parameters and latent state variables, denoted as $p(\Theta|\mathrm{data})$, where $\Theta = (\theta_1, \dots, \theta_k)$ denotes the vector of all of the model parameters and latent state variables, by constructing a Markov Chain that converges to this joint posterior density, while using only the information about the univariate conditional densities $p(\theta_j|\theta_i, i \neq j, \mathrm{data})$, that are far easier to analytically express and sample from.

Multiple types of the MCMC algorithm exist with the most straightforward one being the *Gibbs Sampler*, that can be used to sample form the joint posterior density $p(\Theta|\mathrm{data})$ in the case when we are able to sample from the conditional densities $p\left(\theta_j \middle| \theta_i, i \neq j, \mathrm{data}\right)$.

The Gibbs sampler proceeds as follows:

0. Assign a vector of initial values to $\Theta^0 = (\theta_1^0, \dots, \theta_k^0)$ and set $j = 0$
1. Set $j = j + 1$

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2. Sample $\theta_1^j \sim p(\theta_1|\theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})$
3. Sample $\theta_2^j \sim p\left(\theta_2\middle|\theta_1^j, \theta_3^{j-1}, \dots, \theta_k^{j-1}, \text{data}\right)$

   …
4. Sample $\theta_k^j \sim p(\theta_k|\theta_1^j, \theta_2^j, \dots, \theta_{k-1}^j, \text{data})$ and return to step 1.

As the univariate conditional densities $p(\theta_j|\theta_i, i \neq j, \text{data})$ fully characterize the joint posterior density $p(\Theta|\text{data})$, it can be proved, according to the Clifford-Hammersley theorem [Johannes and Polson, 2009], that the Markov Chain constructed according to the Gibbs Sampler converges to the joint posterior density $p(\Theta|\text{data})$ as its equilibrium density. The distribution of the parameters and the latent state variables of the model can thus be estimated by calculating enough iterations of the Gibbs Sampler, discarding the ones at the beginning, where the algorithm did not converge yet, and using the remaining ones as samples from the joint posterior distribution. The sample mean or median can then be used to estimate the model parameters and latent state variables, while the sample standard deviations can be used to estimate the Bayesian standard errors and test the parameter statistical significance.

The conditional densities $p(\theta_j|\theta_i, i \neq j, \text{data})$ necessarry for the Gibbs sampler construction are typically derived by applying the Bayes theorem to the likelihood function and the prior density. Specifically, the following proportionalle relationship can be utilized:

$$p\left(\theta_1\middle|\theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data}\right) \propto L\left(\text{data}\middle|\theta_1, \theta_2^{j-1}, \dots, \theta_k^{j-1}\right) * \text{prior}\left(\theta_1\middle|\theta_2^{j-1}, \dots, \theta_k^{j-1}\right) \quad (17)$$

with $L(.)$ denoting the likelihood function, $\text{prior}(.)$the Bayesian prior density of the given parameter and $\propto$ the proportionalle relationship. If no prior information is available, the uninformative prior densities, $\text{prior}(\theta_i) \propto 1$, can be used for the prior.

In order to derive the conditional density $p(\theta_j|\theta_i, i \neq j, \text{data})$ for the use in the Gibbs Sampler, it is necessary to normalize the right-hand side of the equation 17, by dividing it with its integral over θ_1, corresponding to the density $p\left(\text{data}\middle|\theta_2^{j-1}, \dots, \theta_k^{j-1}\right)$, thus replacing the proportionalle relationship with equality.

Unfortunately, the integration of the right hand side of equation 17 over θ_1 may often be unfeasible in practice. In such cases the standard Gibbs sampler cannot be used. To construct the MCMC chain it is therefore necessary to use either the so called Rejection Sampling Gibbs Sampler or the Metropolis-Hastings algorithm.

Metropolis-Hastings algorithm is a rejection sampling algorithm, sampling a proposal value of the given parameter (or a latent state) from a proposal density q, and then either accepting or rejecting it, based on a given probability, which leads us to effectively sampling from the conditional density p.

Specifically, to utilize the Metropolis-Hastings algorithm, Step 2 in the Gibbs Sampler algorithm has to be replaced by the following two step procedure:

A. Sample θ_1^j from a proposal density $q(\theta_1|\theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})$

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B. Accept θ_1^j with probability $\alpha = \min(R, 1)$, with R denoting the so called acceptance ratio defined as:

$$R = \frac{p(\theta_1^j | \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data}) q(\theta_1^{j-1} | \theta_1^j, \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})}{p(\theta_1^{j-1} | \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data}) q(\theta_1^j | \theta_1^{j-1}, \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})} \quad (18)$$

Which may in practice be evaluated by sampling $u \sim U(0,1)$ from an uniform distribution and accepting the value of θ_1^j if and only if $u < R$, otherwise the value of the parameter from the previous iteration θ_1^{j-1} is kept instead.

As in the case of the Gibbs Sampler algorithm, it can be shown that the so constructed Markov-chain converges to the joint posterior density $p(\Theta|\text{data})$ as its equilibrium density [Johannes and Polson, 2009].

The computational efficiency of the Metropolis-Hastings algorithm does, however, significantly depend on the choice of the proposal density, which is established based on the specific version of the algorithm

The most simple version of the Metropolis-Hastings algorithm is the *Random-Walk Metropolis-Hastings*, in which the proposal density follows a Random Walk through the parameter space. The proposal q is then defined as:

$$\theta_1^j \sim \theta_1^{j-1} + N(0, c) \quad (19)$$

With c being the step-size meta-parameter which may influence the computational efficiency of the algorithm and the practice is to set it so that approximately 50% of the proposals get accepted and 50% rejected.

A convenient property of the Random-Walk Metropolis-Hastings algorithm is that its proposal distribution is symmetric, in the sense that the probability of going from θ_1^{j-1} to θ_1^j is the same as the probability of going from θ_1^j to θ_1^{j-1} (which is not necessarily true for other types of proposal densities). This causes the terms corresponding to the proposal densities q in the acceptance ratio to cancel out, so the acceptance ratio reduces to:

$$R = \frac{L(\text{data} | \theta_1^j, \theta_2^{j-1}, \dots, \theta_k^{j-1})}{L(\text{data} | \theta_1^{j-1}, \theta_2^{j-1}, \dots, \theta_k^{j-1})} \quad (20)$$

It follows that as long as we are able to calculate the likelihood of the model, it is possible to utilize the Random-Walk Metropolis-Hasting algorithm to estimate the joint posterior density of the model parameters and latent states.

The most time-consuming part of the SVJD model estimation is the sampling of the latent states of the stochastic variances. In our application, this is performed with the *Accept-Reject Gibbs Sampler* algorithm developed by Kim, Shephard and Chib [1998]. As the

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conditional distribution of the stochastic variances $p\left(V_i|\boldsymbol{V}_{(-\boldsymbol{i})},\Theta,\mathbf{r},\mathbf{J},\mathbf{Z}\right)$ cannot be analytically expressed and sampled from, the authors propose to use a proposal distribution q, whose density is at all points above the target density p. The Accept-Reject Gibbs Sampler does then sample in each sampling step repeatedly from q, with the proposal being accepted with an acceptance ratio equal to the ratio of the two densities q/p at the point of the proposal. This has the effect of effectively sampling from the target density p, with the drawback that in every step multiple proposals may have to be sampled from q, until one of them eventually gets accepted.

For the overall proceeding of the algorithm, we are estimating a few model parameters Θ and a large number of latent state variables X. Since we know from the Bayes theorem that:

$$p(\Theta,\mathrm{X}|\mathrm{data}) \propto p(\mathrm{data}|\Theta,\mathrm{X}) * p(\mathrm{X},\Theta) \tag{21}$$

We can estimate iteratively the parameters and the latent states as follows:

$$\begin{aligned} p(\Theta|\mathrm{X},\mathrm{data}) &\propto p(\mathrm{data}|\Theta,\mathrm{X}) * p(\mathrm{X}|\Theta) * p(\Theta) \\ p(\mathrm{X}|\Theta,\mathrm{data}) &\propto p(\mathrm{data}|\Theta,\mathrm{X}) * p(\Theta|\mathrm{X}) * p(\mathrm{X}) \end{aligned} \tag{22}$$

In the employed MCMC algorithm for the SVJD-RV-Z model estimation (i.e. the most complex one out of the proposed models), we have to estimate 13 model parameters $(\mu,\mu_J,\sigma_J,\alpha,\beta,\gamma,\alpha_J,\beta_J,\gamma_J,\sigma_{RV},\mu_Z,\xi_Z,\sigma_Z)$ and 3 vectors of latent state variables $(\boldsymbol{V},\boldsymbol{J},\boldsymbol{Q})$. The algorithm was developed in Fičura and Witzany [2015] and is based on earlier results from Witzany [2013] and is based on the methodology developed in Jacquier et al. [2007] and Johannes and Polson [2009]. The MCMC algorithms for SVJD and SVJD-RV proceed in the same fashion, with slight modifications and some of the steps missing.

The algorithm for SVJD-RV-Z proceeds as follows:

1. Sample initial values of the model latent state variables $\boldsymbol{V}^{(0)},\boldsymbol{J}^{(0)},\boldsymbol{Q}^{(0)}$ and parameters $\mu^{(0)},\mu_J^{(0)},\sigma_J^{(0)},\alpha^{(0)},\beta^{(0)},\gamma^{(0)},\alpha_J^{(0)},\beta_J^{(0)},\gamma_J^{(0)},\sigma_{RV}^{(0)},\mu_Z^{(0)},\xi_Z^{(0)},\sigma_Z^{(0)}$.
2. For $i=1,\dots,T$ sample the jump sizes $J_i^{(g)} \propto \varphi\left(J;\mu_J^{(g-1)},\sigma_J^{(g-1)}\right)$ if $Q_i^{(g)}=0$ using the Gibbs Sampler, or if $Q_i^{(g-1)}=1$, use the Random-Walk Metropolis-Hastings to sample from:
$$J_i^{(g)} \propto \varphi\left(r_i;\mu^{(g-1)}+J,\sqrt{V_i^{(g-1)}}\right)\varphi\left(log(RV_i-J^2);h_i^{(g-1)},\sigma_{RV}^{(g-1)}\right)\varphi\left(J;\mu_J^{(g-1)},\sigma_J^{(g-1)}\right)$$
3. For $i=1,\dots,T$ sample the jump occurrences $Q_i^{(g)}\in\{0,1\}$, using the expression $\Pr[Q=1]=p_1/(p_0+p_1)$, where:
$$p_0=\varphi\left(r_i;\mu^{(g-1)},\sqrt{V_i^{(g-1)}}\right)\varphi\left(\log(RV_i);h_i^{(g-1)},\sigma_{RV}^{(g-1)}\right)\varphi\left(Z_i;\mu_Z^{(g-1)},\sigma_Z^{(g-1)}\right)\left(1-\lambda_i^{(g-1)}\right)$$
$$p_1=\varphi\left(r_i;\mu^{(g-1)}+J_i^{(g)},\sqrt{V_i^{(g-1)}}\right)\varphi\left(\log\left(RV_i-\left(J_i^{(g)}\right)^2\right);h_i^{(g-1)},\sigma_{RV}^{(g-1)}\right)\varphi\left(Z_i;\mu_Z^{(g-1)}+\xi_Z^{(g-1)},\sigma_Z^{(g-1)}\right)\lambda_i^{(g-1)}$$
4. Sample new stochastic log-variances $h_i^{(g)}=\log\left(V_i^{(g)}\right)$ for $i=1,\dots,T$ using the Gibbs Sampler with accept-reject procedure developed by Kim, Shephard and

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Chib [1998], i.e. first calculate the series $y_i = r_i - \mu^{(g-1)} - J_i^{(g)} Q_i^{(g)}$ and then sample $h_i^{(g)}$ from the proposal distribution $\varphi(h_i; \mu_i, \sigma)$, where:

$$\mu_i = \phi_i + \frac{\sigma^2}{2}[y_i^2 \exp(-\phi_i) - 1],$$

$$\phi_i = \frac{\gamma^2 \log\left(RV_i - J_i^2 Q_i\right) + \sigma_{RV}^2[\alpha(1-\beta) + \beta(\log V_{i+1} + \log V_{i-1})]}{\gamma^2 + (1+\beta^2)\sigma_{RV}^2},$$

$$\sigma = \frac{\gamma \sigma_{RV}}{\sqrt{\gamma^2 + (1+\beta^2)\sigma_{RV}^2}}$$

The proposal is accepted with probability f^*/g^*, where:

$$\log(f^*) = -\frac{h_i}{2} - \frac{{y_i}^2}{2}[\exp(-h_i)]$$

$$\log(g^*) = -\frac{h_i}{2} - \frac{{y_i}^2}{2}[\exp(-\phi_i)\,(1+\phi_i) - h_i \exp(-\phi_i)]$$

If no accepted, then another proposal is drawn until acceptance occurs.

5. Sample new stochastic log-variance autoregression coefficients $\alpha^{(g)}, \beta^{(g)}, \gamma^{(g)}$, denoting $h_i = \log\left(V_i^{(g)}\right)$ for $i = 1, \dots, T$, using the Bayesian linear regression model [Lynch, 2007], i.e. define $\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}\boldsymbol{y}$ and $\hat{\boldsymbol{e}} = \boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}$, where $\boldsymbol{X} = \begin{pmatrix} 1 & \dots & 1 \\ h_1 & \dots & h_{T-1} \end{pmatrix}'$ and $\boldsymbol{y} = (h_2 \dots h_T)'$, and sample:

$$\left(\gamma^{(g)}\right)^2 \propto IG\left(\frac{n-2}{2}, \frac{\hat{\boldsymbol{e}}'\hat{\boldsymbol{e}}}{2}\right),$$

$$\left(\alpha^{(g)}, \beta^{(g)}\right)' \propto \varphi\left[(\alpha, \beta)'; \widehat{\boldsymbol{\beta}}, \left(\gamma^{(g)}\right)^2 (\boldsymbol{X}'\boldsymbol{X})^{-1}\right]$$

6. Sample $\mu^{(g)}$ based on the normally distributed time series $r_i - J_i^{(g)} Q_i^{(g)}$ with variances $V_i^{(g)}$ as follows:

$$p\left(\mu^{(g)} | \boldsymbol{r}, \boldsymbol{J}^{(g)}, \boldsymbol{Q}^{(g)}, \boldsymbol{V}^{(g)}\right) \propto \varphi\left(\mu; \sum_{i=1}^{T} \frac{r_i - J_i^{(g)} Q_i^{(g)}}{V_i^{(g)}} \Big/ \sum_{i=1}^{T} \frac{1}{V_i^{(g)}}, \sum_{i=1}^{T} \frac{1}{V_i^{(g)}}\right)$$

7. Sample the Hawkes process parameters $\theta_J, \beta_J, \gamma_J$, using the Random-Walk Metropolis-Hastings algorithm with the proposal densities given as:

$$\theta_J^{(g)} = \theta_J^{(g-1)} + N(0, c),$$

$$\beta_J^{(g)} = \beta_J^{(g-1)} + N(0, c),$$

$$\gamma_J^{(g)} = \gamma_J^{(g-1)} + N(0, c),$$

and the likelihood function equal to $L\left(\boldsymbol{Q}^{(g)} | \theta_J, \beta_J, \gamma_J\right) = \prod_{i=1}^{T} \lambda_i^{Q_i} (1-\lambda_i)^{1-Q_i}$.

8. Sample $\mu_J^{(g)}, \sigma_J^{(g)}$ based on the normally distributed time series $\boldsymbol{J}^{(g)}$ and uninformative priors $p(\mu) \propto 1$ and $p(\log\sigma^2) \propto 1$, equivalent to $p(\sigma^2) \propto 1/\sigma^2$:

$$p\left(\mu_J^{(g)} | \boldsymbol{J}^{(g)}, \sigma_J^{(g-1)}\right) \propto \varphi\left(\mu_J^{(g)}; \frac{\sum_{i=1}^{T} J_i^{(g)}}{T}, \frac{\sigma_J^{(g-1)}}{\sqrt{T}}\right)$$

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$$p\left[\left(\sigma_J^{(g)}\right)^2 |\boldsymbol{J}^{(g)}, \mu_J^{(g)}\right] \propto IG\left[\left(\sigma_J^{(g)}\right)^2; \frac{T}{2}, \frac{\sum_{i=1}^{T}\left(J_i^{(g)} - \mu_J^{(g)}\right)^2}{2}\right]$$

9. Sample $\sigma_{RV}^{(g)}$ using the Inverse Gamma density:

$$p\left[\left(\sigma_{RV}^{(g)}\right)^2 |\boldsymbol{RV}, \boldsymbol{J}^{(g)}, \boldsymbol{Q}^{(g)}, \boldsymbol{V}^{(g)}\right] \propto IG\left[\left(\sigma_{RV}^{(g)}\right)^2; \frac{T}{2}, \frac{\sum_{i=1}^{T}\left(\log\left(RV_i - \left(J_i^{(g)}\right)^2 Q_i^{(g)}\right) - h_i^{(g)}\right)^2}{2}\right]$$

10. Sample $\mu_Z^{(g)}, \xi_Z^{(g)}, \sigma_Z^{(g)}$ using the normally distributed series $Z_i - Q_i^{(g)}\xi_Z^{(g-1)}$, with variance $\sigma_Z^{(g-1)}$ to sample $\mu_Z^{(g)}$, series $Q_i^{(g)}\left(Z_i - \mu_Z^{(g)}\right)$, at points where $Q_i^{(g)} = 1$, with variance $\sigma_Z^{(g-1)}$, to sample $\xi_Z^{(g)}$, and the centralized time series $Z_i - \mu_Z^{(g)} - Q_i^{(g)}\xi_Z^{(g)}$ to sample $\sigma_Z^{(g)}$ using the Inverse Gamma distribution.

The implementation of the MCMC algorithm used for the estimation of the SVJD-RV-Z model in Matlab can be found in the Appendix.

The MCMC algorithm enables us to estimate the posterior distribution of the model parameters and latent state variables conditional on the full data sample. Specifically, for a time series of length T, we are estimating the distribution of model parameters and latent sate variables $p(\Theta, \mathrm{X}|\mathcal{F}_T)$, conditional on the filtration $\mathcal{F}_T$, denoting the available information up until time T (i.e. the full time series). While this may be useful in order to fit the model to the data, it is impractical in the case of forecasting and especially in the case of back-testing of the models out-of-sample forecasts of volatility and jumps, as it would require the re-estimation of the MCMC algorithm for each time point in the past, in order to get the parameter and latent state estimates as they were available at the given time point (i.e. conditional on the observable information $\mathcal{F}_t$, with $t = 1, \dots T$).

As this would be too time-consuming, a common approach is to combine the MCMC algorithm with Particle Filters [Fulop, Li and Yu, 2015], with the MCMC used in order to estimate the model parameters and latent state variables in the in-sample period, while the Particle Filters are used to sequentially estimate the evolution of the latent state variables in the out-sample period, conditional on the parameters estimated in the in-sample period, and the filtrations $\mathcal{F}_t$ at every time point in the out-sample part of the time series.

Particle Filters (also known as Sequential Monte-Carlo algorithm) are Bayesian estimation methods, using a weighted set of particles, together with Bayesian recursion equations, in order to sequentially estimate the posterior densities of the latent state variables of an econometric model for each time-point in the time series, conditional on the observable information up to the given time-point.

Specifically, assuming that we have an observable series y_t of a length T, governed by a set of known model parameters Θ and a latent state time series x_t, the MCMC algorithm estimates for each time point t the posterior distribution of the corresponding latent state $p(x_t|\mathcal{F}_T, \Theta)$, with $\mathcal{F}_T$ denoting the filtration (i.e. available information about the evolution of y_t) up until the end of the dataset T. The Particle Filter algorithm, on the other hand,

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estimates for each time point t the posterior distribution $p(x_t|\mathcal{F}_t, \Theta)$, which is conditional only on the observable information up until the time t.

The Particle Filter represents each distribution $p(x_t|\mathcal{F}_t)$ with a weighted set of particles, which can then be used, together with the model parameters Θ, to estimate the distributions $p(x_{t+1}|\mathcal{F}_t)$, $p(x_{t+2}|\mathcal{F}_t)$ via simulations of the future evolution of the latent state variables. In our study, we first estimate the model parameters Θ and the in-sample evolution of the latent state variables on the in-sample period with an MCMC algorithm, and then use these estimates to initialize and run a *Sequential Importance Resampling (SIR) Particle Filter* to sequentially estimate the evolution of the latent state variables in the out-sample period.

To define the SIR Particle Filter, we use an illustrative example with y_t denoting observations of the observable time series and x_t the observations of the latent time series, that is governing the dynamics y_t. The purpose of the SIR Particle Filter is to estimate the posterior distribution of x_t at each time point, conditional on the evolution of x_t up to that time point and the known parameters of the model Θ.

To estimate the posterior distribution of x_t at each time point t, the SIR Particle Filter uses a set of $L = 1, \dots, P$ weighted particles $\mathrm{x}_t^{(L)}$, each of them representing one possible path of x_t. Bayesian recursion equations are then used to adjust the weights of the particles $w_t^{(L)}$ in order to represent the posterior distribution $p(x_t|y_0, \dots, y_t)$. The values of the particles $x_t^{(L)}$ and their weights $w_t^{(L)}$ can then be used to approximate the expectation of any desired function of x_t (such as mean, median or standard deviation), as follows:

$$\int_{-\infty}^{\infty} f(x_t) p(x_t|y_0, \dots, y_t) dx_t \approx \sum_{i=1}^{P} x_t^{(i)} f\left(x_t^{(i)}\right) \tag{23}$$

The SIR Particle Filter introduces a re-sampling phase into the Particle Filter algorithm in order to avoid the problem of degeneracy of the particles, which is a situation in which all of the weights become very close to zero except for one.

The SIR Particle Filter algorithm proceeds as follows:

1. For $L = 1, \dots, P$ particles draw samples from the proposal density $x_t^{(L)} \sim \pi\left(x_t|x_{0:t-1}^{(L)}, y_{0:t}\right)$
2. For $L = 1, \dots, P$ update the importance weights up to a normalizing constant according to:

$$w_t^{*(L)} = w_{t-1}^{(L)} \frac{p\left(y_t|x_t^{(L)}\right) p\left(x_t^{(L)}|x_{t-1}^{(L)}\right)}{\pi\left(x_t^{(L)}|x_{0:t-1}^{(L)}, y_{0:t}\right)} \tag{24}$$

A common approach is to use proposal density equal to the conditional density of x_t, in this case $\pi\left(x_t^{(L)}|x_{0:t-1}^{(L)}, y_{0:t}\right) = p\left(x_t^{(L)}|x_{t-1}^{(L)}\right)$, the weight updating equation simplifies then to:

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$$w_t^{*(L)} = w_{t-1}^{(L)} p\left(y_t | x_t^{(L)}\right) \quad (25)$$

Where $p\left(y_t | x_t^{(L)}\right)$ corresponds to the likelihood of y_t conditional on $x_t^{(L)}$

3. For $L = 1, \dots, P$ compute the normalized weights: $w_t^{(L)} = \frac{w_t^{*(L)}}{\sum_{J=1}^{P} w_t^{*(J)}}$
4. Compute effective number of particles: $N_{eff} = \frac{1}{\sum_{J=1}^{P}\left(w_t^{(J)}\right)^2}$
5. If $N_{eff} < N_{thr}$ then resample the particles with probabilities proportional to their weights and for $L = 1, \dots, P$ set $w_t^{(L)} = 1/P$

In our case, the observable time series are the past returns $r(t)$ and the past realized variances $\mathrm{RV}(t)$ and values of the Z-Statistics $Z(t)$, in the case of the SVJD-RV and SVJD-RV-Z models respectively. The unobservable time series are the stochastic log-variances $h(t)$ (or stochastic variances $\mathrm{V}(t)$ alternatively), the jump occurrences $Q(t)$ and the jump sizes $J(t)$. The jump intensities $\lambda(t)$, although also latent, do not need to be estimated separately, as they deterministically depend on the past jump occurrences $Q(t)$.

1.5 Simulation study of the in-sample model fit

A simulation study is performed, in order to assess the ability of the SVJD, SVJD-RV and SVJD-RV-Z models to estimate stochastic volatility and jumps in simulated time series with different jump magnitudes. The accuracy of the model estimates is then compared with the accuracy of the non-parametric estimators of integrated variance (using the EIV estimator) and jumps (using the Z-Estimator).

First, high-frequency (15-minute) time series are simulated with different values of the σ_J parameter determining the absolute size of the simulated jumps. Daily returns and daily values of the power-variation estimators are computed from the simulated time series and the SVJD, SVJD-RV and SVJD-RV-Z models are estimated on them, using the MCMC algorithm described in the previous section.

The model estimates of latent stochastic variances are then compared with the daily integrated variances derived from the simulations and the model latent state time series of jump occurrences are compared with the jump occurrences in the simulations. The accuracy of the model estimates is then compared with the accuracy of the non-parametric estimators of the integrated variances (EIV) and jumps (Z-Estimator).

In order to assess, how well do the estimated stochastic variances $E(V_i|model)$ fit to the daily integrated variances from the simulations, the R-Squared criterion is used.

To assess the ability of the models to identify jumps in the simulated time series, the Accuracy Ratio (Gini Coefficient) is applied to the daily time series of estimated jump occurrences and the daily time series of jumps computed from the simulations by assuming that as long as at least one jump occurred during the given day, the value of $Q_i = 1$.

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To define the *Accuracy Ratio (AR)*, we have to first define $p_{success}$ as the probability of a successful discrimination, defined in a sense that if $Q_{i,Jump}$ denotes a random day at which a jump occurred and $Q_{j,NoJump}$ a random day at which no jump occurred, $p_{success}$ represents the probability that $E(Q_i|model) > E(Q_j|model)$, with $E(Q_i|model)$ denoting the probability of jump occurrence assigned by the model to day $Q_{i,Jump}$ and $E(Q_j|model)$ the probability of jump occurrence assigned by the model to day $Q_{j,NoJump}$. Similarly, we define p_{fail} as the probability, that for two randomly chosen days $Q_{i,Jump}$ and $Q_{j,NoJump}$, defined as above, the opposite holds, i.e. $(Q_i|model) < E(Q_j|model)$.
Using the above probabilities $p_{success}$ and p_{fail}, it is possible to calculate the Accuracy Ratio (AR) as follows:

$$AR = p_{success} - p_{fail} \tag{26}$$

In the performed study, altogether 8 high-frequency (15-minute) time series of 5000 days (480 000 15-minute periods) are simulated, with the value of σ_J varying in the simulations between $\sigma_J = 0.0025$ and $\sigma_J = 0.02$ (with the increment of 0.0025), so that the average absolute jump size ranges from 0.25% to 2%.
The rest of the parameters used in the simulation was set to empirically realistic values, as estimated as in the study of Fičura and Witzany [2015] in which a SVJD model was fitted to the EUR/USD time series. The parameters are shown in Table 1:

Table 1 | Parameters of the SVJD model used in the simulation study

	mui	muiJ	sigmaJ	alpha	beta	gamma	lambdaLT	betaJ	gammaJ
Parameter	0.0001	0.0000	x	-0.0475	0.9954	0.0686	0.0205	0.4414	0.0423

The muiJ parameter was set to zero, so that the jumps have a zero mean, and the sigmaJ parameter varies between 0.0025 and 0.02 as mentioned above.
The simulated daily returns and stochastic variances (or more specifically integrated variances, as they were aggregated to the daily frequency) for 4 of the simulated time series (with sigmaJ alternatively 0.5%, 1%, 1.5% and 2%), are shown on Figure 1 and Figure 2. Figure 3 does then shows the daily realized variances computed from the simulated 15-minute returns.
As can be seen from Figure 1, the simulated stochastic variances (daily integrated variances) exhibit long-term clustering, which is caused by the high value of the beta parameter used in simulations (beta=0.995), which is typical for the empirically observed financial time series. The magnitude of the simulated returns on Figure 1 does clearly correspond to the simulated variances on Figure 2. The presence of jumps in the simulations is apparent in the return plot (Figure 1) and even more clearly in the realized variance plot (Figure 3), especially in the case of the higher values of sigmaJ.

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Figure 1 | Daily logarithmic returns of the simulated time series

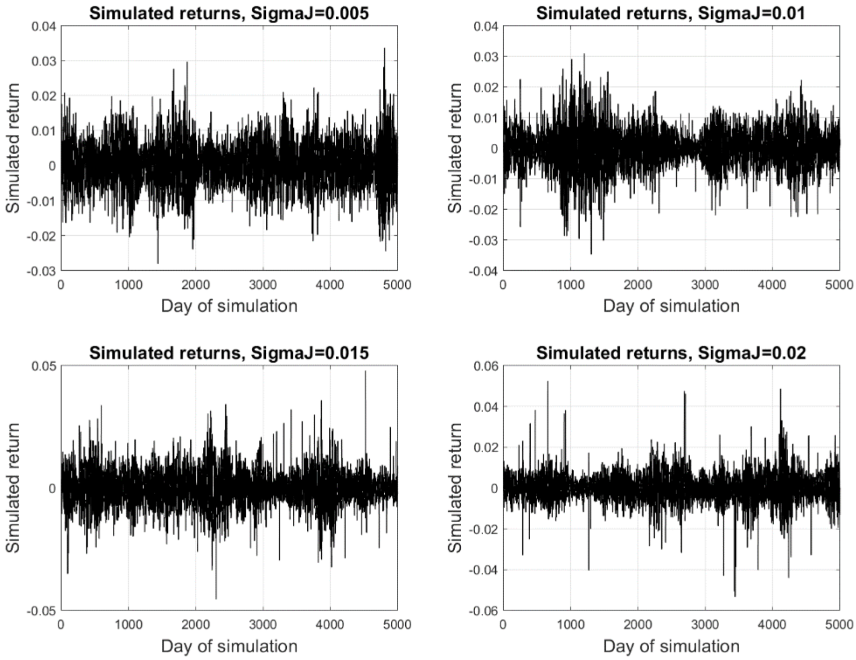

Figure 2 | Daily integrated variances of the simulated time series

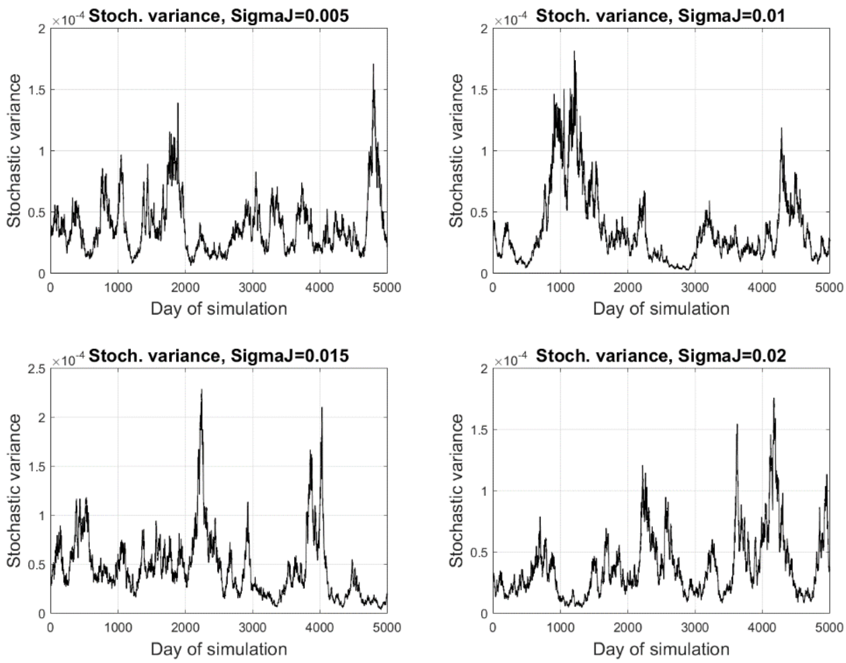

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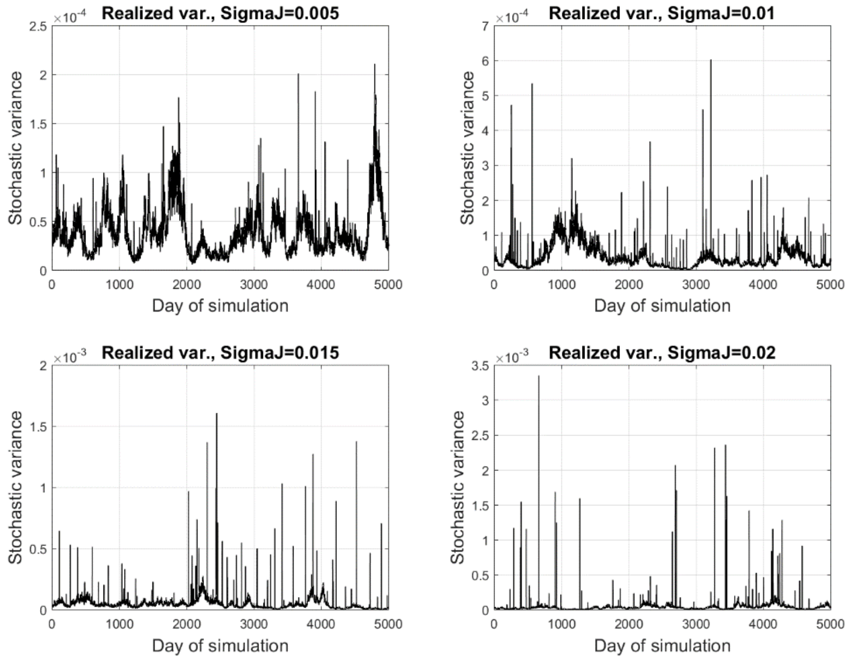
Figure 3 | Daily realized variances of the simulated time series



From the 8 simulated time series of intraday returns, we first computed the daily returns, realized variances and the values of the Z-Statistics for every day. These were then used to estimate the SVJD, SVJD-RV and SVJD-RV-Z models, by using a MCMC algorithm with 5 000 iterations, with the first 2 000 discarded and the remaining 3 000 used for parameter and latent state estimation based on the posterior means.

By calculating the posterior means of the estimated latent time series of jump occurrences Q_t, we get Bayesian estimates of the probabilities of jump occurrence for every day in the time series [Witzany, 2013]. These are then compared with the daily jump occurrences derived from the simulations and Accuracy Ratios of the estimates are computed.

The results are shown in Table 2, with the first 3 columns corresponding to the Accuracy Ratios of the Bayesian jump estimates of the 3 parametric models (SVJD, SVJD-RV and SVJD-RV-Z), while the last column measures the accuracy of the non-parametric jump estimates constructed by applying the Gaussian cumulative distribution function to the values of the Z-Estimator.

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Table 2 | Accuracy ratios of the jump estimates of the models, when applied to simulated time series with different values of SigmaJ

SigmaJ	SVJD	SVJD-RV	SVJD-RV-Z	Z-Est
0.0025	0.0195	0.4757	0.2409	0.3166
0.005	0.2300	0.6622	0.4528	0.5643
0.0075	0.3292	0.7757	0.6135	0.7295
0.01	0.4697	0.8250	0.6860	0.7927
0.0125	0.5274	0.8476	0.7728	0.8341
0.015	0.5905	0.9353	0.7931	0.8860
0.0175	0.5633	0.8131	0.7199	0.8566
0.02	0.6394	0.8686	0.7550	0.8403

We can see from the results in Table 2, that the most accurate jump estimates were in most of the simulations achieved by the SVJD-RV model, which outperformed both, the SVJD-RV-Z model as well as the non-parametric Z-Estimator based approach. The worse performance of the SVJD-RV-Z model can be explained by the fact that the employed approach divides the jumps in the time series into "small jumps" and "large jumps", with the latent time series estimated by the model modelling only the "large jumps" (i.e. the ones having an impact on the overall daily returns), which might have influenced the results for this model. The ability of the SVJD-RV model to outperform the Z-Estimator approach is, however, unexpected and represents a potentially interesting result for practical applications as the Z-Estimator is commonly used in practice as benchmark. Finally, the SVJD model proved to be the least accurate for jump identification, which is to be expected, as it is the only model that does not use the information from the intraday data.

Table 3 shows the accuracy of the daily stochastic variance (integrated variance) estimates of the models, measured with the R-Squared statistics. The last column does in this case correspond to the accuracy of the non-parametric EIT estimator of integrated variance, defined as in equation 11, with the confidence level alpha equal to 0.95.

Table 3 | R-Squared of the integrated variance estimates of the models, when applied to simulated time series with different values of SigmaJ

SigmaJ	SVJD	SVJD-RV	SVJD-RV-Z	EIV
0.0025	0.8547	0.9886	0.9887	0.9516
0.005	0.8197	0.9830	0.9831	0.9152
0.0075	0.8206	0.9860	0.9859	0.9266
0.01	0.8865	0.9898	0.9899	0.9435
0.0125	0.8094	0.9843	0.9839	0.9066
0.015	0.8681	0.9849	0.9846	0.9087
0.0175	0.8201	0.9831	0.9840	0.9145
0.02	0.8535	0.9844	0.9845	0.8702

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In Table 3 we can see the accuracy of the model integrated variance estimates, measured by the R-Squared criterion. As we can see, the accuracy of the SVJD-RV and SVJD-RV-Z models is comparable in this case and it is also in general higher than of the SVJD model and even of the non-parametric EIV based approach. This is again an important result as it shows that the SJVD models extended with the power-variation estimators, do provide more accurate estimates of the underlying stochastic variance than the purely non-parametric approach that is often used in practice. Additionally, we can see that while the accuracy of the EIV based approach drops with the increasing value of SigmaJ, no significant drop can be observed for the parametric models, indicating that they handle the jump component in the data better than the non-parametric method. The standard SVJD model, estimated on daily returns, did again achieve the lowest accuracy out of the tested models, which could be expected as it does not use the additional information contained in the intraday returns.

1.6 Empirical application to foreign exchange

In the performed empirical study, we apply the presented SVJD, SVJD-RV and SVJD-RV-Z models to the time series of 4 major foreign exchange rates, namely to EUR/USD, GBP/USD, USD/CHF and USD/JPY, with the dataset ranging from 1.11.1999 to 15.6.2015, containing altogether 4 072 daily observations.

In the first part of the analysis, the models are estimated on the full data sample, using the MCMC method, in order to assess the properties of the stochastic variances and jumps estimated with the 3 tested models.

In the second part of the analysis, out-of-sample predictive power of the models is evaluated, by first estimating them on the first 2 000 days of the time series (in-sample period ranging from 1.11.1999 to 3.7.2007), followed with a sequential estimation of the latent state variables evolution over the out-sample period (4.7.2007 to 15.6.2015), using the SIR Particle Filter algorithm. At each time point, forecasts of the 1-day, 5-day and 20-day ahead quadratic variations are calculated and compared with the future realized variances, in order to assess the model predictive power. The realized variances are in this case used as a proxy of the future quadratic variations as we cannot observe them directly from the data. The realized variance and Z-Estimator time series for the estimation of the SVJD-RV and SVJD-RV-Z model, and finally for the forecast accuracy assessment, were calculated from the intraday 15-minute return time series, provided by Forexhistorydatabase.com.

All of the calculations (MCMC and Particle Filters) were performed in Matlab, with the major scripts (for the MCMC and the SIR Particle Filter corresponding to the SVJD-RV-Z model) available in the Appendix.

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1.6.1 Full-Sample Analysis

The SVJD, SVJD-RV and SVJD-RV-Z models were applied to the daily return time series of EUR/USD, GBP/USD, USD/CHF and USD/JPY over the history ranging from 1.11.1999 to 15.6.2015.

Figure 4 shows the evolution of the prices of the 4 analysed currency exchange rates. Figure 5 shows the evolution of their logarithmic returns, Figure 6 shows the evolution of the realized variances and Figure 7 shows the evolution of the Z-Statistic (with both, the realized variances and the Z-Statistic calculated from 15-minute returns).

It is apparent from the return plot (Figure 5) and the realize variance plot (Figure 6) that the exchange rates exhibit time varying volatility with long-term volatility clusters, especially around the crisis period of 2009. Large values of the Z-Statistics (i.e. significantly larger than zero) do further indicate that the analysed time series contain large number of jumps, with the most prominent ones being visible even on the daily return and realized variance plots (especially the jump associated with the end of the monetary interventions of the Swiss central banks that happened in 2015 in the USD/CHF exchange rate), while most of the small jumps (i.e. the values of the Z-Statistics significantly larger than zero) cannot be directly observed from the return and realized variance plots as they are lost in the overall price variability.

In the next step, a MCMC algorithm with 5 000 iterations was used to fit the SVJD, SVJD-RV and SVJD-RV-Z models to the observed time series (of returns, realized variances and the Z-Statistics), with the first 2 000 iterations discarded (as the algorithm might have not converged to the posterior distribution yet) and the remaining ones then used for the model parameter and latent state estimation based on the posterior means, as well as for the parameter standard error estimation based on the posterior standard deviations.

Convergence of the MCMC algorithm for selected parameters of the SVJD-RV-Z model, when applied to the EUR/USD exchange rate, is shown on the Figure 8 (muiJ, sigmaJ, alpha and beta), Figure 9 (gamma, lambdaLT, betaJ and gammaJ) and Figure 10 (sigmaRV, muiZ, ksiZ and sigmaZ). The lambdaLT is in this case equal to the thetaJ parameter in the previous section and corresponds to the long-term jump intensity.

We can see from the plots, that most of the parameters converged rather quickly to the joint posterior distribution, within the first several hundred iterations (as is apparent for example from the gamma parameter of the log-variance process).

The convergence is less clear only for the betaJ parameter of the Hawkes process determining the rate of decay of the jump intensities. Nevertheless, when a histogram is computed from the betaJ parameter values over the last iterations, it has a significant mode in the upper region of the parameter space, indicating a convergence (although with a strong left-tail of the posterior distribution, indicating a potentially low statistical significance of the estimate, as will be further seen from the parameter standard errors).

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Figure 4 | Evolution of the daily price of the 4 analysed foreign exchange rates

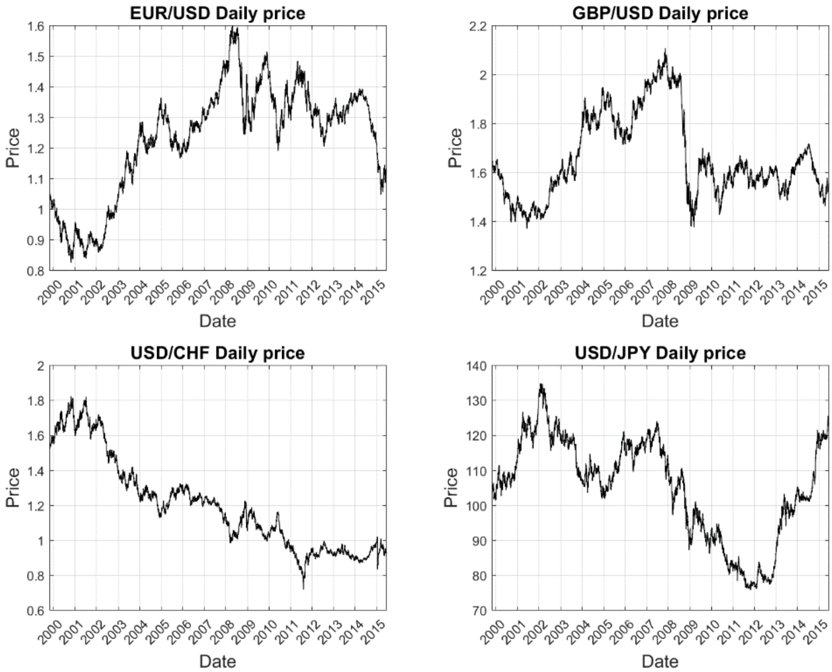

Figure 5 | Evolution of the daily returns of the 4 analysed foreign exchange rates

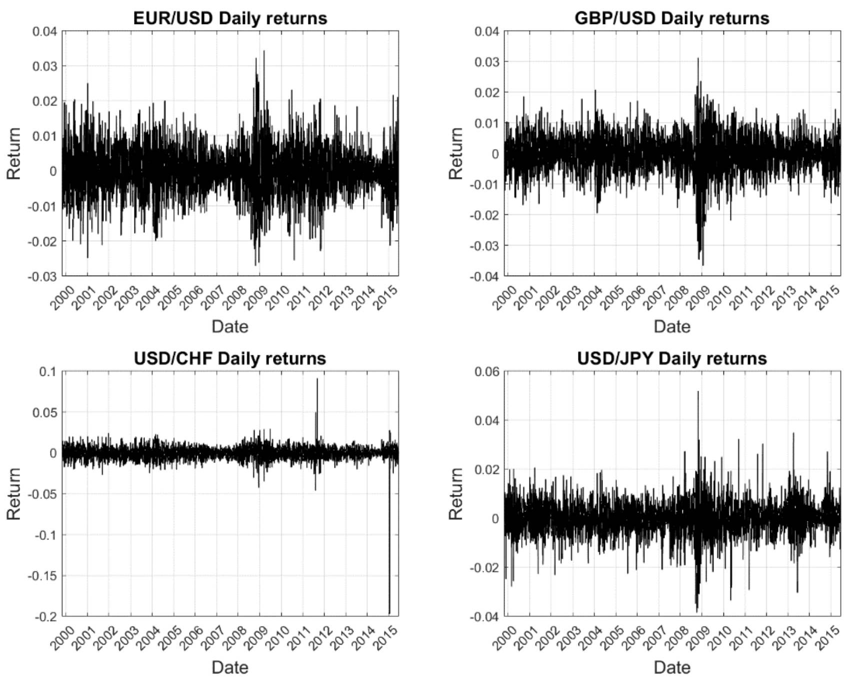

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Figure 6 | Evolution of the realized variance of the 4 analysed foreign exchange rates

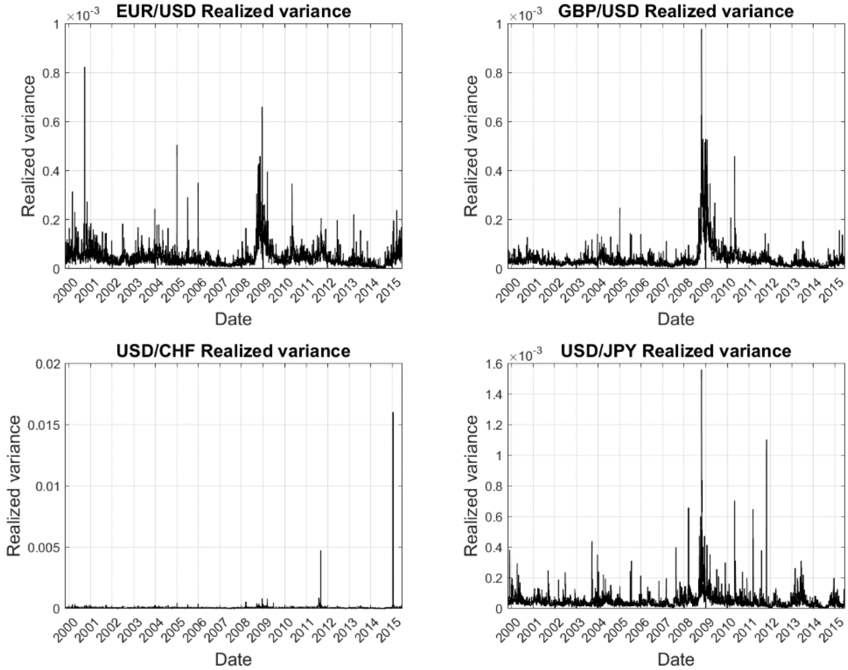

Figure 7 | Evolution of the Z-Estimator of the 4 analysed foreign exchange rates

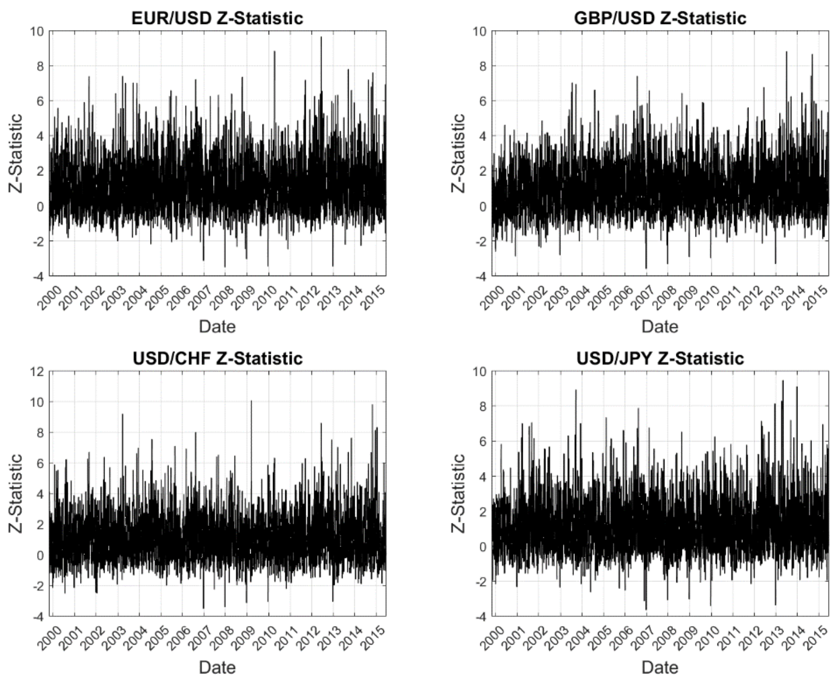

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Figure 8 | Convergence of the MCMC algorithm for the SVJD-RV-Z model (EUR/USD)

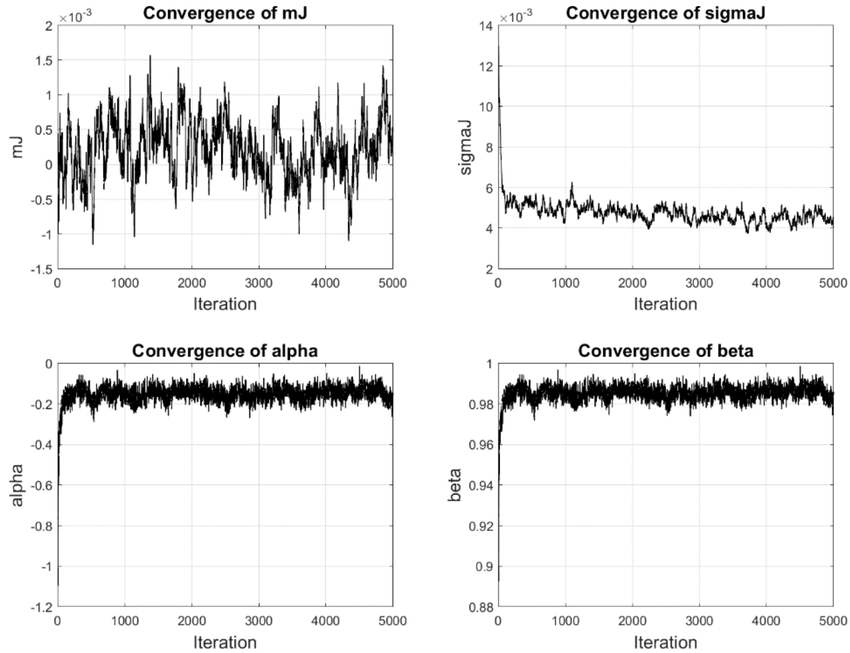

Figure 9 | Convergence of the MCMC algorithm for the SVJD-RV-Z model (EUR/USD)

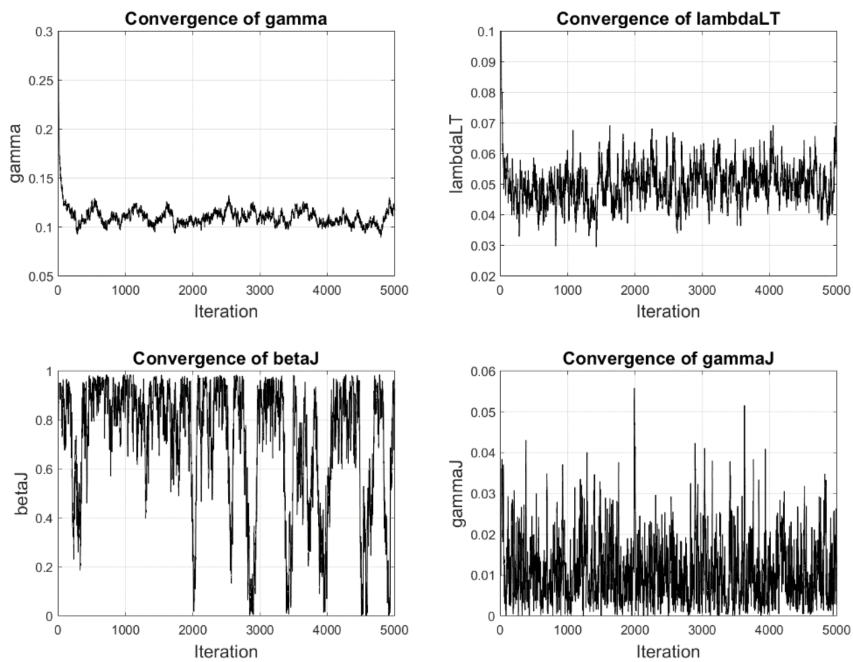

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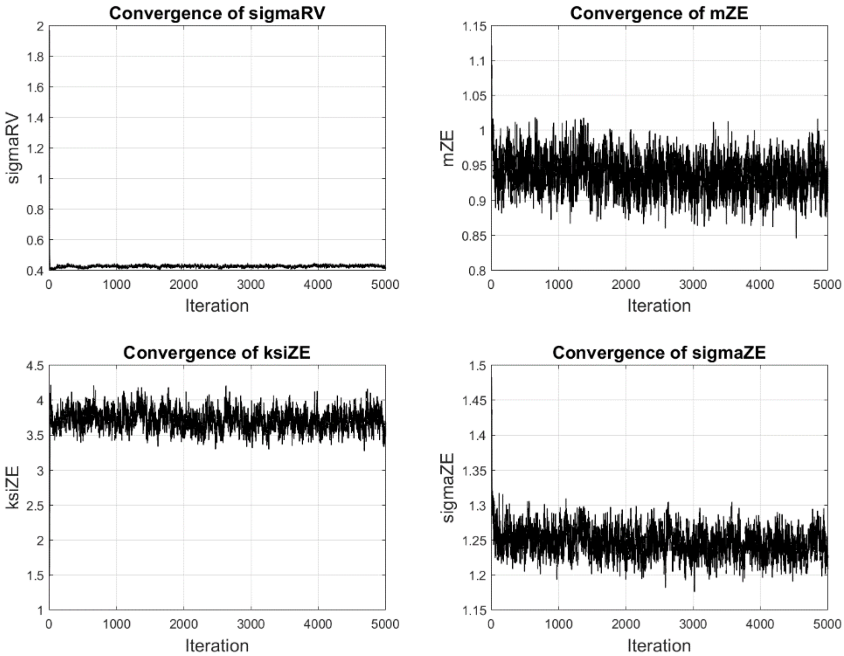
Figure 10 | Convergence of the MCMC algorithm for the SVJD-RV-Z model (EUR/USD)



The final MCMC parameter estimates (based on the posterior means) and the Bayesian standard errors (based on the posterior standard deviations) for the tested models (SVJD, SVJD-RV and SVJD-RV-Z), applied to the 4 analysed currencies, are shown in Table 4 (EUR/USD), Table 5 (GBP/USD), Table 6 (USD/CHF) and Table 7 (USD/JPY).

The evolution of the estimated latent state variable time series (i.e. of the stochastic variances and of the Bayesian jump probabilities of occurrence, computed as an average of the jumps occurrences in the MCMC iterations) is then depicted on Figures 11-18.

From the parameter estimates in the Tables 5-7, we can see that in all of the cases is the value of beta, determining the log-variance process persistence, very close to one, indicating a highly persistent process. At the same time is the beta in all of the cases significantly (i.e. more than two standard errors) lower than one, indicating that the log-variance process is still stationary (although highly persistent).

The value of sigmaJ, determining the jump magnitude, was generally estimated to be far higher for the SVJD and SVJD-RV models than in the case of the SVJD-RV-Z model. This is to be expected, as the SVJD-RV-Z uses the additional information from the Z-Statistics in the jump estimation, enabling it to pick even the smaller jumps in the time series, which the other two models (using only the information from the daily returns and realized variances) cannot distinguish from the continuous volatility component.

The lower values of sigmaJ in the case of the SVJD-RV-Z model (compared to the other models) coincide with its much higher values of the lambdaLT parameter (i.e. the thetaJ), representing the mean long-term jump intensity. The SVJD-RV-Z model does thus seem

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to identify larger number of jumps in the time series than the other two models, due to its utilization of the Z-Statistics, which enables it to pick also the smaller jumps, that the other two models cannot identify.

The estimated parameters of the Hawkes process (betaJ and gammaJ) show that most of the time series (and models) do not exhibit statistically significant jump clustering, as the value of gammaJ, representing the degree of jump self-excitation (i.e. how much the jump intensity increases in the day following a jump occurrence) is in most of the cases less than 2 standard errors away from one and thus statistically insignificant.

An exception can be seen in the estimates of the SVJD-RV model for the CHF/USD time series for which the gammaJ is statistically significant, but betaJ is not, indicating a co-jump behaviour, when a jump occurrence increases the probability of an additional jump in the following day, but not so much in the days afterwards. A similar co-jump behaviour can be observed also for the SVJD-RV model applied to the JPY/USD, although with a lesser statistical significance.

A more persistent jump clustering (although with a relatively weak statistical significance of the gammaJ) can be observed for the SVJD-RV-Z model and the GBP/USD and EUR/USD currencies, for which is the betaJ parameter significant. Nevertheless, even in these cases is the value of betaJ much smaller than one, indicating a rather low duration of the clustering effect, lasting for at most a few days.

In the case of the SVJD-RV-Z model, we can further observe that the value of muiZ was in all of the cases estimated as significantly larger than zero (usually around 1), indicating a presence of small jumps in the high-frequency time series of the returns, which shift the value of the Z-Statistics upwards (compared to the no-jump case of 0). The large jumps, having an impact on the daily returns, do then further increase the values of the Z-Statistics by additional 3-4 points, as can be seen from the values of ksiZ.

The large number of small intraday jumps is then apparent also from the latent state time series of jump occurrences, corresponding to the Bayesian probabilities of jump occurrence. As we can see for example from Figure 12 (for EUR/USD), the SVJD-RV-Z model identified a significantly higher number of probable jumps in the time series than the SVJD-RV and SVJD models. At the same time, however, is the number of identified jumps much smaller than what would be implied by the values of the Z-Estimator alone (as can be seen from the lower-right subplot). The difference in the number of jumps on the lower two subplots shows how numerous the small intraday jumps in fact are.

From the time series of the Bayesian probabilities of jump occurrence we can further see that the standard SVJD model was not able to identify almost any jumps with a high probability of occurrence, indicating that the estimation of jumps solely from the daily returns is problematic.

The series of the estimated stochastic variances (Figure 11 for EUR/USD) show us that the stochastic variance estimates of the SVJD-RV and SVJD-RV-Z models are more variable than the estimates of the basic SVJD model, but less variable than the realized variance itself, indicating that the extended SVJD models are able to filter the noise plaguing the in the realized variance estimates of the stochastic variance, at least to a certain degree.

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Table 4 | Full-sample MCMC parameter estimates and std. errors for the EUR/USD

	EUR/USD					
	SVJD		SVJD-RV		SVJR-RV-Z	
	Estimate	Std.Err	Estimate	Std.Err	Estimate	Std.Err
mui	0.0001	0.0001	0.0001	0.0001	0.0000	0.0001
muiJ	0.0076	0.0052	-0.0021	0.0023	0.0002	0.0004
sigmaJ	0.0102	0.0012	0.0109	0.0015	0.0046	0.0003
alpha	-0.0466	0.0169	-0.1498	0.0339	-0.1480	0.0323
beta	0.9955	0.0016	0.9855	0.0033	0.9857	0.0031
gamma	0.0627	0.0045	0.1099	0.0080	0.1094	0.0067
lambdaLT	0.0061	0.0050	0.0087	0.0030	0.0510	0.0059
betaJ	0.4888	0.2866	0.4214	0.2552	0.6555	0.2892
gammaJ	0.0377	0.0254	0.0483	0.0284	0.0104	0.0076
sigmaRV			0.4252	0.0065	0.4266	0.0060
muiZ					0.9339	0.0243
ksiZ					3.6992	0.1356
sigmaZ					1.2432	0.0184

Table 5 | Full-sample MCMC parameter estimates and std. errors for the GBP/USD

	GBP/USD					
	SVJD		SVJD-RV		SVJR-RV-Z	
	Estimate	Std.Err	Estimate	Std.Err	Estimate	Std.Err
mui	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
muiJ	-0.0004	0.0009	-0.0015	0.0028	0.0000	0.0004
sigmaJ	0.0057	0.0005	0.0082	0.0014	0.0041	0.0004
alpha	-0.0852	0.0240	-0.1157	0.0271	-0.1182	0.0285
beta	0.9920	0.0022	0.9890	0.0026	0.9888	0.0027
gamma	0.0806	0.0067	0.0922	0.0048	0.0942	0.0055
lambdaLT	0.0698	0.0161	0.0057	0.0022	0.0427	0.0083
betaJ	0.4665	0.2588	0.3423	0.2514	0.7545	0.3067
gammaJ	0.0260	0.0211	0.0410	0.0274	0.0160	0.0091
sigmaRV			0.3949	0.0052	0.3874	0.0052
muiZ					0.7826	0.0260
ksiZ					3.5542	0.1641
sigmaZ					1.2151	0.0206

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Table 6 | Full-sample MCMC parameter estimates and std. errors for the USD/CHF

	USD/CHF					
	SVJD		SVJD-RV		SVJR-RV-Z	
	Estimate	Std.Err	Estimate	Std.Err	Estimate	Std.Err
mui	0.0000	0.0001	-0.0001	0.0001	-0.0001	0.0001
muiJ	-0.0019	0.0014	0.0212	0.0218	0.0006	0.0013
sigmaJ	0.0082	0.0012	0.0491	0.0130	0.0098	0.0016
alpha	-0.2463	0.0671	-0.2258	0.0438	-0.3148	0.0538
beta	0.9759	0.0066	0.9778	0.0043	0.9690	0.0053
gamma	0.1510	0.0167	0.1201	0.0082	0.1455	0.0092
lambdaLT	0.0603	0.0258	0.0036	0.0014	0.0275	0.0048
betaJ	0.4537	0.2571	0.3586	0.2402	0.5536	0.2946
gammaJ	0.0298	0.0237	0.0604	0.0236	0.0161	0.0140
sigmaRV			0.4229	0.0060	0.4119	0.0062
muiZ					0.9354	0.0246
ksiZ					4.3716	0.2289
sigmaZ					1.3120	0.0204

Table 7 | Full-sample MCMC parameter estimates and std. errors for the USD/JPY

	USD/JPY					
	SVJD		SVJD-RV		SVJR-RV-Z	
	Estimate	Std.Err	Estimate	Std.Err	Estimate	Std.Err
mui	0.0001	0.0001	0.0002	0.0001	0.0002	0.0001
muiJ	-0.0014	0.0012	-0.0017	0.0035	0.0003	0.0007
sigmaJ	0.0106	0.0013	0.0141	0.0028	0.0068	0.0004
alpha	-0.1607	0.0383	-0.6797	0.0946	-0.7286	0.0968
beta	0.9847	0.0036	0.9341	0.0092	0.9295	0.0094
gamma	0.1121	0.0119	0.2255	0.0142	0.2363	0.0138
lambdaLT	0.0557	0.0193	0.0095	0.0034	0.0414	0.0054
betaJ	0.4402	0.2584	0.4057	0.2534	0.3524	0.2422
gammaJ	0.0297	0.0234	0.0533	0.0274	0.0117	0.0102
sigmaRV			0.4008	0.0083	0.3872	0.0089
muiZ					0.9573	0.0242
ksiZ					3.7210	0.1489
sigmaZ					1.2923	0.0191

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Figure 11 | MCMC estimated latent series of stochastic variances (EUR/USD)

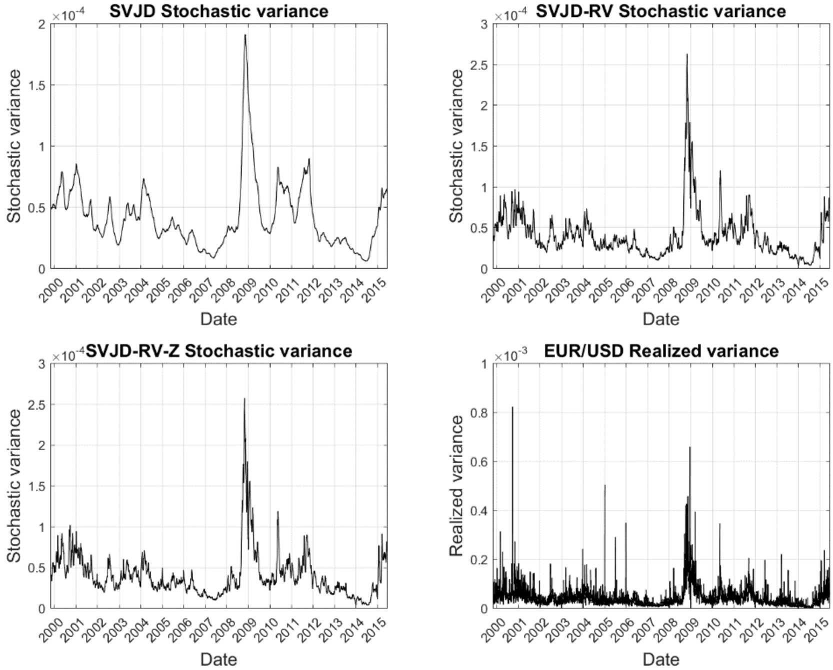

Figure 12 | MCMC estimated probabilities of jump occurrences (EUR/USD)

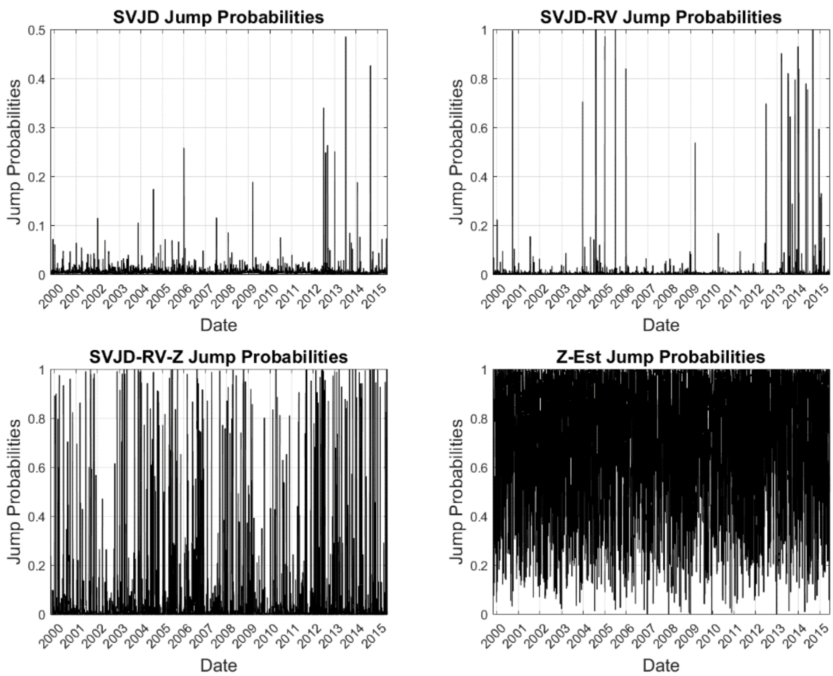

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Figure 13 | MCMC estimated latent series of stochastic variances (GBP/USD)

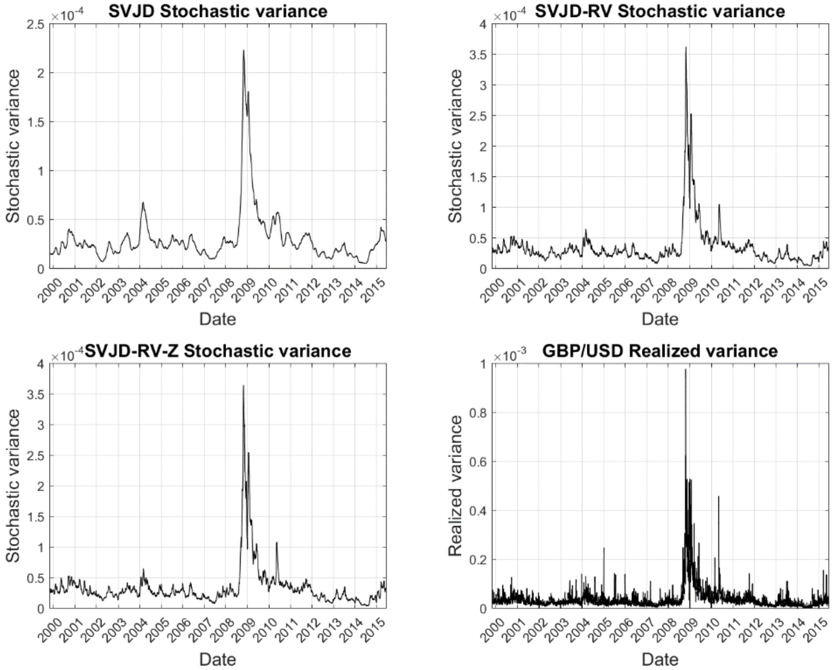

Figure 14 | MCMC estimated probabilities of jump occurrences (GBP/USD)

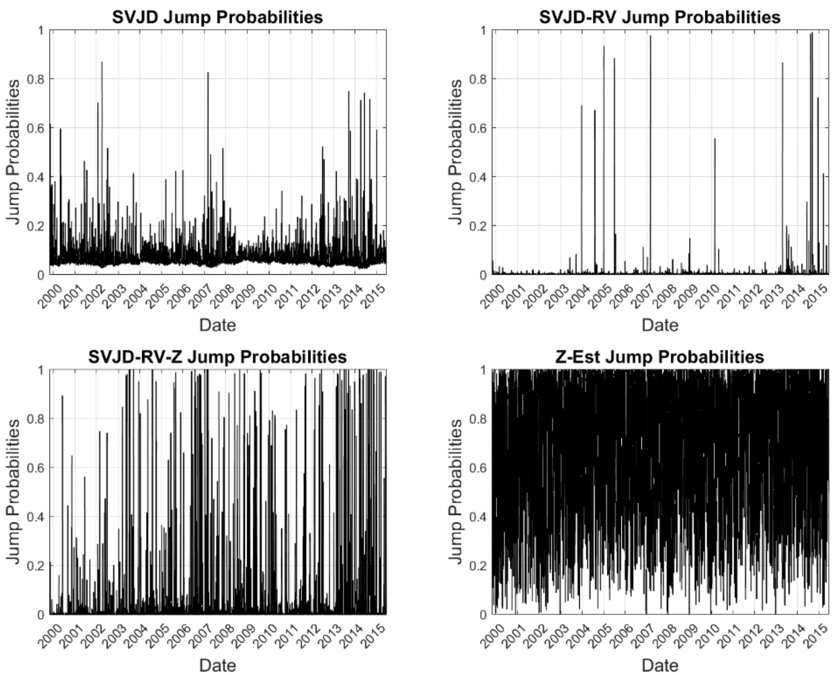

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Figure 15 | MCMC estimated latent series of stochastic variances (USD/CHF)

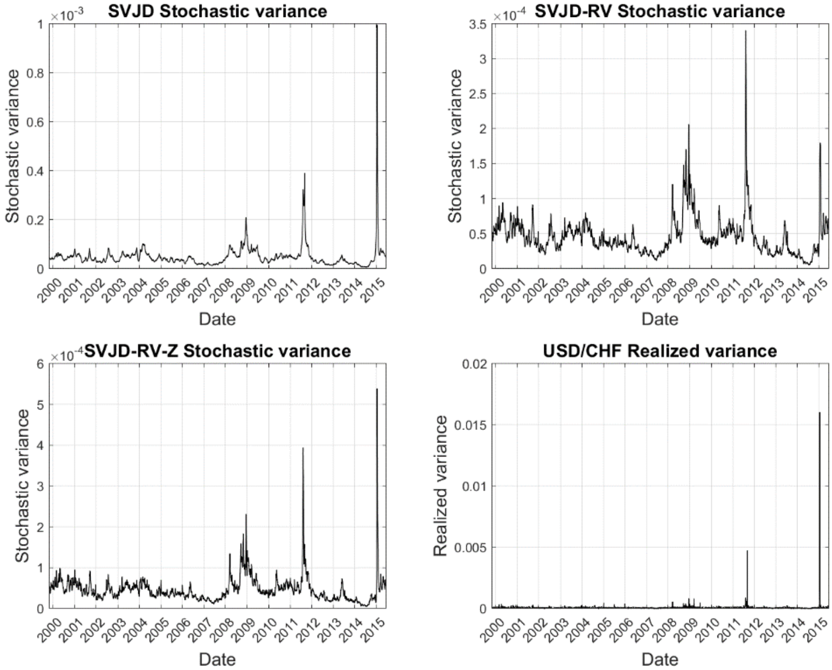

Figure 16 | MCMC estimated probabilities of jump occurrences (USD/CHF)

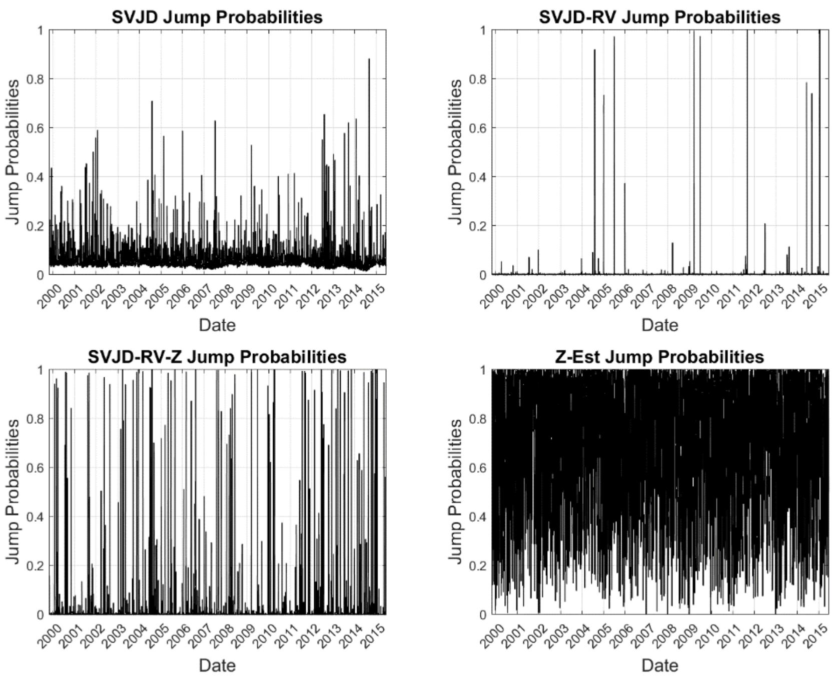

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Figure 17 | MCMC estimated latent series of stochastic variances (USD/JPY)

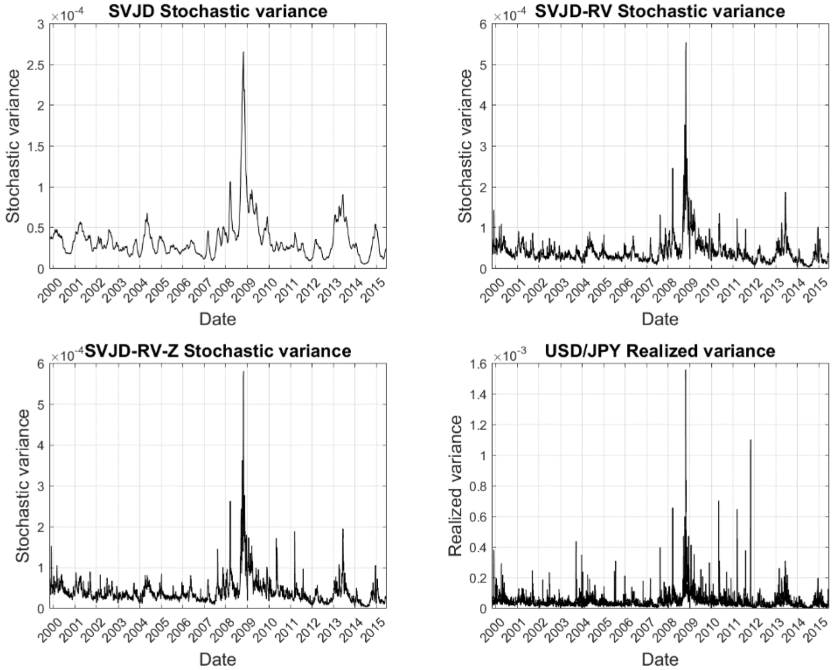
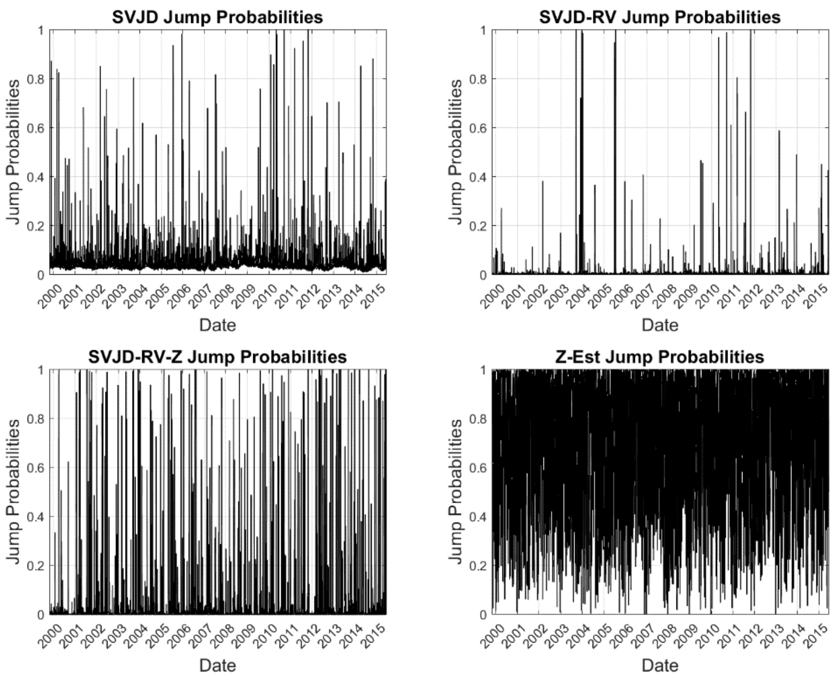


Figure 18 | MCMC estimated probabilities of jump occurrences (USD/JPY)



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1.6.2 Out-Sample Analysis

For the out-sample analysis of the predictive accuracy of the models (SVJD, SVJD-RV and SVJD-RV-Z), the foreign exchange rate time series are split into an in-sample period (first 2 000 days ranging from 1.11.1999 to 3.7.2007) and an out-sample period (4.7.2007 to 15.6.2015). The MCMC algorithm is then applied to the in-sample period to estimate the model parameters and the in-sample evolution of the latent state variables (the stochastic variances, jump occurrences and jump sizes). SIR Particle Filter is then used to sequentially estimate the evolution of the latent state variables in the out-sample period.

This enables us to estimate the posterior distributions of the latent states at each time point in the out-sample period, conditional on the observed history until the given time point. Predictions into the future are then generated by sampling the latent states (i.e. particles of the particle filter) at each time point and simulating their evolution into the future by using the model equations. The goal of the study is to assess the predictive accuracy of the models with regards to the future quadratic variance.

MCMC algorithm is used for the in-sample estimation of the model parameters and latent state variables. The algorithm is run over 5 000 iterations, with the first 2 000 discarded and the latter 3 000 used to calculate the parameter estimates based on the posterior means. The SIR Particle Filter used to sequentially estimate the evolution of the latent state variables over the out-sample period is run with 20 000 particles, whose values are initialized by sampling from the MCMC estimated distribution of the latent states in the last day of the in-sample period. The threshold for re-sampling was set to 200 particles.

At each time-point (i.e. day) of the out-sample period, simulations of the future evolution of the particles are run, to simulate the evolution of the stochastic variances and jumps in a 1-day, 5-day and a 20-day horizon. These are then used to calculate the quadratic variation for each simulation and the simulated quadratic variations are then averaged to arrive at the mean forecast of the quadratic variation in the given forecast horizon.

The forecasts of the quadratic variations are compared with the future realized variance (computed for a 1-day, 5-day or 20-day period), which is used as a proxy of the unobserved quadratic variation in the future. R-Squared values of the forecasts are then computed. Table 8 contains the results of the out-sample test.

Table 8 – Out-Sample R-Squared of the quadratic variation forecasts

	1-Day horizon			5-Day horizon			20-Day horizon		
	SVJD	SVJD-RV	SVJD-RV-Z	SVJD	SVJD-RV	SVJD-RV-Z	SVJD	SVJD-RV	SVJD-RV-Z
EUR/USD	0.4506	0.5576	0.5576	0.5827	0.6840	0.6772	0.5715	0.6709	0.6574
GBP/USD	0.3235	0.5832	0.5945	0.3582	0.6484	0.6503	0.2734	0.5053	0.5249
USD/CHF	0.0067	0.0077	0.0081	0.0182	0.0246	0.0240	0.0348	0.0502	0.0509
USD/JPY	0.1600	0.2442	0.2827	0.2149	0.2744	0.2950	0.1994	0.1942	0.1748

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From the results in Table 8 we can see that the SVJD-RV and SVJD-RV-Z models achieve in most of the cases similar predictive accuracy, while the predictive power of the standard SVJD model is lower. This is in line with the results of the previous analyses and it indicates that it is indeed worthy to utilize the power-variation estimators when estimating SVJD models. The SVJD-RV-Z model outperforms the SVJD-RV model especially at the 20-day horizon for the EUR/USD and GBP/USD exchange rate and on the 1-day horizon for the USD/JPY exchange rate, the differences are, however, rather small.

1.7 Conclusions

A methodology was presented of how to utilize power-variation estimators computed from high-frequency data in the Bayesian estimation of Stochastic-Volatility Jump-Diffusion (SVJD) models. A standard SVJD model uses only the information from daily returns to estimate the latent stochastic variances, jump occurrences and jump sizes of the time series. In the presented study, two extended models were proposed, the SVJD-RV model, utilizing additionally the realized variance estimator for the estimation of the stochastic variance, and the SVJD-RV-Z model, utilizing in addition to that the non-parametric Z-Estimator, to estimate the jump occurrences in the time series.

A MCMC algorithm, combining a Gibbs Sampler and a Metropolis Hastings algorithm was presented, in order to estimate the parameters and the latent state variable time series of the proposed models in the in-sample period, while the Sequential Importance Resampling (SIR) Particle Filter was proposed, to sequentially estimate the evolution of the latent states in the out-sample period and use it to generate forecasts via simulations.

In the simulation part of the study, the proposed models were estimated on simulated time series with different jump magnitudes and the accuracy of the model fit to the underlying stochastic variances and jumps was assessed. It was shown that both of the proposed models utilizing the power-variation estimators clearly outperform the standard SVJD model that uses only the daily data. The SVJD-RV model achieved the best results in the simulation study, outperforming the SVJD-RV-Z model, as well as the non-parametric approaches for the estimation of the stochastic variances and jumps in the simulated series.

Empirical study of the models was performed in order to assess their out-sample predictive accuracy with regards to the future quadratic variances. In the first step, the models were fitted to the in-sample evolution of 4 foreign exchange time series (EUR/USD, GBP/USD, USD/JPY and USD/CHF) by using a MCMC algorithm. SIR Particle Filter was then used to sequentially estimate the evolution of the latent state variables (i.e. stochastic variances, jump occurrences and jump sizes) over the out-sample period, and simulations were used to calculate 1-day, 5-day and 20-day forecasts of the process quadratic variation. The forecasts were then compared with the subsequently observed realized variance. The out-sample predictive accuracy of the SVJD-RV and SVJD-RV-Z models, using the power variation estimators, was higher than in the case of the simple SVJD model that works only with the daily returns.

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