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DIPLOMA THESIS

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Dynamic Portfolio Optimization During Economic Recession

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Abstract: When working with portfolio optimization, we encounter problems with the proper way of representing the risk measure and choosing the suitable optimization method. These problems are even more prominent in times of economic recession, such as the global COVID pandemic, when the markets are extremely volatile. The aim of the thesis is to describe the main approaches to the modelling of the market risk using Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models family. Day-to-day dynamic portfolio optimization approaches are subsequently utilized, and adaptations of Markowitz Model and Mean-ES model are compared. Adaptations of Markowitz model are based on replacing the diagonal of the covariance matrix of the considered assets with differently estimated various risk measures such as volatility, or Expected Shortfall (ES), estimated with GARCH and Exponential GARCH (EGARCH) models. Comparison of the individual approaches is based on their returns during the first half-year of the global COVID pandemic, for a suitably selected portfolio. The difference between approaches using Markowitz model showed to be only minor. Therefore it can not be deduced that one approach is better than the other, it can be only assumed that it is better to utilize ES, rather than volatility, as a risk measure during the economic recession. However, the best approach to portfolio optimization during the economic recession was seemed to be Mean-ES where there was an immense difference in return.

Keywords: Expected Shortfall, Value at Risk, GARCH models, Markowitz model, Dynamic portfolio optimization, Sharpe Ratio

Contents

1. Introduction

Economic recessions occur over time for different reasons. The Great Depression in the 1930s was caused by a money supply reduction and therefore a banking crisis. The Great Recession observed from 2007 to 2009, began with the bursting of the United States housing bubble in 2005 to 2006. However, the latest economic recession, which could be observed worldwide began by a totally different cause.

On December 31, 2019, according to World Health Organization (WHO) Wuhan Municipal Health Commission, China, reported a cluster of cases of pneumonia in Wuhan, Hubei Province. A novel coronavirus was eventually identified. The first recorded case outside of China was confirmed on January 13, 2020, in Thailand. By the end of January, with 7818 confirmed cases across 19 countries, WHO declared the COVID-19 outbreak. Most countries undertook serious strategic plans to slow the spread of the virus by partially closing their economies and closing their borders. This moderately helped and slowed the spread but it was not a long-term solution. Since then the spread of the virus was still accelerating as countries reopened their economies, although many countries have made progress in slowing down the spread (Feuer and Kim, 2020). This made countries restrict their economies again. All this led to the worst economic downturn since The Great Depression (Gopinath, 2020).

Especially during such times, it is essential to be able to optimize the portfolio efficiently to avoid large losses. Optimization portfolio theory examined the biggest breakthrough when Harry Markowitz firstly presented his model in 1952 (Markowitz, 1952) and has examined the blossom since then. Markowitz model considers the risk of the given portfolio as a variance of its returns. What makes it easily understandable while keeping computational complexity low. Since 1952, a lot of time has passed, new technologies were developed and the computing power of computers arose significantly. These developments made an opportunity to practice computational complex approaches to portfolio optimization as well. As an example of such a computationally intensive task, we can present Mean-Expected Shortfall (Rockafellar and Uryasev, 2000). Moreover, in this thesis, we work with dynamic portfolio optimization. We run optimization models on daily basis similarly as in Blasques (2017), where for each day multiple optimization models with different constraints run, creating an efficient frontier.

To choose only one resulting portfolio for a given day Sharpe Ratio proposed in Sharpe (1994) is used.

All of the approaches to the portfolio optimization mentioned before are working with some risk measure. The basic definitions of risk measures can be found in multiple sources such as McNeil et al. (2005), Artzner et al. (1999), or Danielsson (2011). The most common ways to represent the risk of a market are by its volatility, Value at Risk (VaR), or Expected Shortfall (ES). VaR became more popular when it was in 1996 presented by J.P. Morgan and Reuters as their measure of risk (Morgan, 1996). Since then it became contained in the Basel II framework as well (on Banking Supervision, 2005). Many more publications were written on this topic such as Jorion (2006), Danielsson (2011), or Roccioletti (2016). However, VaR has some drawbacks which were overcome by Expected Shortfall - ES. ES was firstly proposed by Artzner et al. (1999) and a comparison with VaR can be found in Roccioletti (2016).

Now when we have presented the risk measures the question lies in the way how to model them. This thesis covers the parametric models represented by the generalized autoregressive conditional heteroskedasticity (GARCH) models family. GARCH models were firstly presented by Bollerslev (1986) and Taylor (1986) in the late eighties and are built on autoregressive conditional heteroskedasticity models (ARCH) (Engle, 1982). This evolution of the models is described in Cipra (2008). In this thesis, we cover GARCH model, EGARCH model proposed by Nelson (1991), and CCC-Garch proposed by Bollerslev (1990) and described in Cipra (2008).

This thesis's purpose lies in describing the main approaches to the modelling of the market risk using models family. Followed by day-to-day dynamic portfolio optimization and comparing the returns of the individual approaches during the first half-year of the global COVID pandemic on a suitably selected portfolio. In Section 2.1 risk measures VaR and ES are introduced. Followed by Sections 2.2 and 2.3, where ways to model these measures are presented. In Sections 2.4, 2.5 and 2.6 optimization models are introduced. Finally, these approaches of dynamic portfolio optimization are in the empirical part of this thesis in Chapter 4 applied on given portfolios using suitably chosen data, consisting of daily closing prices on New York Stock Exchange of selected S&P 500 stocks (described in Chapter 3), during COVID-19 pandemic and the returns of these portfolios are examined.

2. Research Methodology

2.1 Risk Measures

Throughout this thesis, we work with terms derived from the word root "Risk". Since the term "Risk" has ambiguous meaning and it is very difficult to pin down precisely, to define it we can turn to Oxford Learner's Dictionary.

Definition 1 (Risk). *Risk is the possibility of something bad happening at some time in the future; a situation that could be dangerous or have a bad result.*

However, in this thesis, we work with market risk, one of the most important risks in banking. The market risk is connected to some instruments or a portfolio of instruments we own. These instruments are usually having volatility over time, which results in possible losses and therefore risk. For a more specific definition, we can use McNeil et al. (2005), where other risks connected to banking, such as credit risk and operational risk, are mentioned as well.

Definition 2 (Market Risk). *Market Risk is the risk of a change in the value of a nancial position due to changes in the value of the underlying components on which that position depends, such as stock and bond prices, exchange rates, commodity prices, etc.*

During the times it has been shown that knowing more information about the risk connected to our instruments or portfolio, can make us higher profit or in the other case at least minimize the loss. We can look at this problem as knowing when it is the right time to undertake the risk and continue to own the portfolio or sell it instead. In these terms, there were developed methods such as risk measures.

Given some "reference instrument," there is a natural way to define a measure of risk by describing how close or how far from accepting a position is (Artzner et al., 1999). Based on Artzner et al. (1999) we can define the basic risk measure. Let Ω be the set of all-natural states. We assume it is finite. Let $\mathcal{G}$ be the set of all risks, that is the set of all real-valued functions on Ω. Based on Roccioletti (2016) they are representing the final net worth of some instruments or portfolio of instruments for each element of Ω.

Definition 3 (Risk measure). *A risk measure $\rho(X)$ is a mapping from $\mathcal{G}$, set of all risks, into $\mathbb{R}$.*

In other words from Danielsson (2011) risk measure is a mathematical method for computing risk. Similarly, we can define risk measurement based on Danielsson (2011).

Definition 4 (Risk measurement). *Risk measurement is a number that captures risk. It is obtained by applying data to a risk measure.*

This way we can express the riskiness of a position with only one number. The riskier a position is, the higher its measure of risk will be. It can be considered as an aid for a decision-maker, either it is regulator or supervisor, who decides rather he does accept the position with the given risk or not.
Like that we can generate an acceptance set $\mathcal{A}$, the set of all the positions with given risk that are acceptable by decision-maker.

Definition 5 (Acceptance set $\mathcal{A}$). *A set that consists of nal net worths, that are accepted by decision-maker. Acceptance set $\mathcal{A}$ holds that the acceptance set $\mathcal{A}$ contains all of the non negative elements in $\mathcal{G}$.*

Based on Artzner et al. (1999) we can define Acceptance set $\mathcal{A}$, associated with a risk measure.

Definition 6 (Acceptance set $\mathcal{A}$, associated with a risk measure). *The acceptance set associated with a risk measure ρ is the set denoted $\mathcal{A}_\rho$ and de ned by*

$$\mathcal{A}_\rho = \{X \in \mathcal{G} | \rho(X) \leq 0\}.$$

Till today, several Risk measures have been already developed. The most popular and common are realized Volatility, Value at Risk (VaR), Expected Shortfall (ES), and one of the latest using Expectiles. However, they are using different principles and it happens, that for the same instrument or portfolio of instruments they return different values. When they return the same one or a very similar one we can choose the risk measure that is the simplest to implement, however when they do not return the same values it is up to us to decide which one to use. Even nowadays it is still not straightforward which risk measure we should use. The goal of today's research is to find such a measure that will be robust and perform well for most of the cases. In the following parts, we will focus on VaR and ES considering multiple approaches in their calculation.

2.1.1 Value at Risk

Value at Risk (VaR) is one of the most widely used risk measures in the financial sector. It has become the classic measure used by financial executives to quantify the market risk. In June 1996 J.P. Morgan and Reuters teamed up on RiskMetrics and announced VaR as its measure of risk (Morgan, 1996). Later on, in 2002, VaR was also contained in the Basel II capital-adequacy framework on Banking Supervision (2005), which later on brought some controversy, and this way financial institutions were enforced to meet capital requirements based on VaR estimates.

Now we will provide a formal definition based on McNeil et al. (2005). Let's consider a portfolio of risky assets and a fixed time horizon Δ. Next, let's suppose that we have estimated the loss distribution[1] to this portfolio. We can denote its distribution function by $F_L(l) = P(L \leq l)$. We can define VaR as follows.

Definition 7 (Value at Risk–VaR). *Given some con dence level $\alpha \in [0, 1]$. The VaR of the portfolio at the con dence level α is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1 - \alpha)$. Mathematically,*

$$VaR_\alpha = inf\{l \in \mathsf{R} : P(L > l) \leq 1 - \alpha\} = inf\{l \in \mathsf{R} : F_L(l) \geq \alpha\}.$$

In statistical terms, we can define the VaR as the quantile of the loss distribution. The most typical values for α are $\alpha = 0.95$ or $\alpha = 0.99$. On the other hand, the most typical values for time horizon Δ, during which portfolio is held unchanged, considered for a bank's trading desk is a one day and ten business days for calculation of capital requirements, in credit risk management and operational risk management Δ is usually one year.

Example 1 (VaR for Gauss loss distribution). *Let's suppose that the loss distribution F_L is Gaussian with mean μ and standard deviation σ. Then $L \sim$*

[1] Practitioners are often involved with the so-called profit-and-loss (P&L) distribution. However, in risk management, we are mainly concerned with the probability of large losses and hence we often drop the profits from P&L. Moreover, since actuarial risk theory is a theory of positive random variables, we will use the convention that losses are positive numbers and we will focus on the upper (right) tail of the loss distribution L. (Roccioletti, 2016)

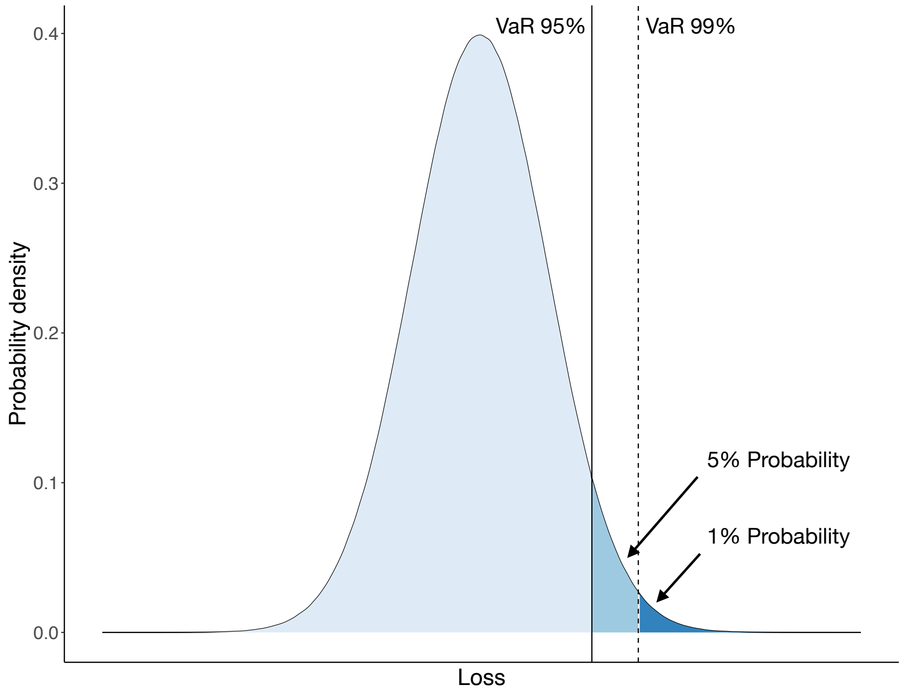

Figure 2.1: An example of a loss distribution with the 95% VaR marked as a vertical line and and 99% VaR marked with a dashed line.

$F_L(\mu, \sigma^2)$. *Next we consider* α *xed,* $\alpha \in [0, 1]$. *Then*

$$P(L \leq \mathit{VaR}_\alpha) = P\left(\frac{L-\mu}{\sigma} \leq \frac{\mathit{VaR}_\alpha - \mu}{\sigma}\right) = P\left(\frac{L-\mu}{\sigma} \leq \Phi\ ^{1}(\alpha)\right)$$

$$= \Phi\left(\Phi\ ^{1}(\alpha)\right) = \alpha,$$

where Φ *denotes the Gauss distribution function and* $\Phi\ ^{1}(\alpha)$ *is the* α*-th quantile of* Φ. *Using this we can de ne VaR as follows*

$$\mathit{VaR}_\alpha(L) = \mu + \sigma\Phi\ ^{1}(\alpha). \tag{2.1}$$

As McNeil et al. (2005) mentioned this result is often used in variance-covariance approach also known as the delta-normal approach.

As a demonstrative example, we can look at Figure 2.1, where L distribution with VaR values is shown.

Another interesting statistic of a portfolio based on distribution to this portfolio is a maximum possible loss, given by $inf\{l \in \mathsf{R} : F_L(l) = 1\}$, which is according to McNeil et al. (2005) important in reinsurance. However, usually, the F_L is unbounded so that the maximum loss can reach infinity therefore in this case it does not have any probability information.

Despite the commonness and popularity of this risk measure, VaR has its drawbacks. The two important drawbacks are lack of sub-additivity and being tail insensitive. By the lack of sub-additivity, we mean that VaR does not award the benefits of diversification. And by being tail insensitive we mean that despite knowing that the loss will be in $\alpha(100\%)$ of cases lower than a certain level, we do not know anything about the size of the loss in remaining $(1-\alpha)100\%$ of cases. The critique of the first one has its origins in the work of Artzner et al. (1997, 1999), who showed that the VaR is not a coherent[2] risk measure because it is believed that reasonable risk measures should contain the sub-additivity. The other authors joined this fundamental critique, such as Taleb (1997), who wrote: "VaR is charlatanism because it tries to estimate something that is scientifically impossible to estimate, namely the risk of rare events. It gives people a misleading sense of precision."

These drawbacks were more exposed during the global financial crisis in 2007-2008 when VaR underestimated the risk, so financial institutions were undertaking much higher risk. The controversy was caused mainly because VaR was still contained in Basel II capital-adequacy framework on Banking Supervision (2005). In response the Basel Committee proposed three major changes to the existing regime in 2013, to be incorporated into Basel III on Banking Supervision (2010). The most important was the replacement of 99% VaR with 97.5% ES, which is more sensitive to loss distribution in the tail of the distribution. We will focus on ES in the next chapter.

2.1.2 Expected Shortfall

It took a while to find a coherent alternative to VaR (Roccioletti, 2016). A natural choice turned out to be Expected Shortfall[3] (ES), which is closely connected to VaR and was proposed by Artzner et al. (1999). Nowadays it is preferred to VaR by many risk practitioners because of overcoming drawbacks

[2] As coherent measures are meant measures following logical and consistent principles. To read more about coherent rules connected to VaR see Artzner et al. (1999).

[3] Expected Shortfall is also known as Conditional Value at Risk (CVaR), Average Value at Risk (AVaR), and Expected Tail Loss (ETL).

of VaR. The most important one is that ES is a coherent measure, therefore always satisfies the subadditivity, compared to VaR which does not. It seems that rather than asking what is the minimum loss in presupposed percentage of the worst cases of losses, it is a better idea to ask what is the expected value of the losses within that presupposed percentage of the worst cases of losses. The ES is the answer to this question and tells us what is the expected loss when VaR is exceeded or in other words, the expected value of the losses with probability exceeding a given confidence level.

Definition 8 (Expected Shortfall–ES). *Let L be the loss function with $E(|L|) < \infty$ and F_L its distribution function (df). Then given some con dence level $\alpha \in [0,1]$, the ES of our portfolio at the con dence level α is de ned as*

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)du,$$

where q_u is the quantile function of F_L.

Definition 9 (Expected Shortfall in relation with Value at Risk). *Let L be the loss function with $E(|L|) < \infty$. Then given some con dence level $\alpha \in [0,1]$, the ES of our portfolio at the con dence level α is de ned as*

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du.$$

As Roccioletti (2016) stated, this is principal and most used formulation of ES.

Example 2 (ES for Gaussian loss distribution). *Let's suppose that the loss distribution F_L is Gaussian with mean μ and standard deviation σ. Then $L \sim F_L(\mu, \sigma^2)$. Next we consider α xed, $\alpha \in [0,1]$. Then*

$$ES_\alpha = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha},$$

where ϕ denotes the probability density of Gaussian distribution and $\Phi^{-1}(\alpha)$ is the α-th quantile of Φ.

Similarly, we can derive the formula for ES for Student t loss distribution, see McNeil et al. (2005).

As a demonstrative example, we can look at Figure 2.2, where L distribution with 95% VaR value and 95% ES value is shown. Clearly, ES is higher than VaR.

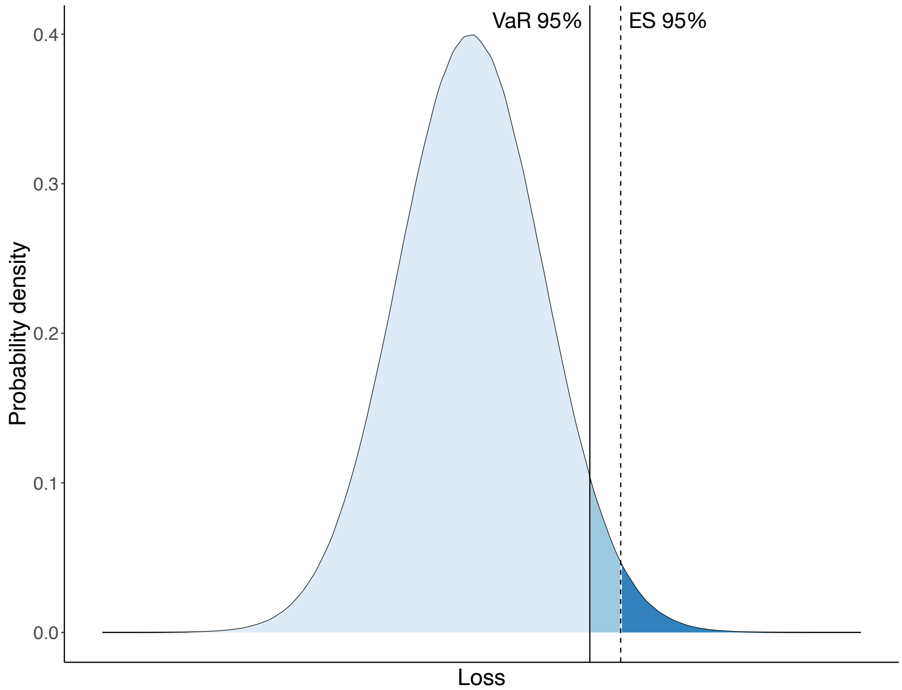

Figure 2.2: An example of a loss distribution with the 95% VaR marked as a vertical line and and 95% ES marked with a dashed line.

Although ES overcomes the drawbacks of VaR it still has some weaknesses. In particular, ES is not an elicitable[4] risk measure. However, the Basel Committee incorporated ES into Basel III, where 97.5% ES replaces 99% VaR.

2.2 Modelling VaR and ES

VaR is a very intuitive concept. However, its correct estimation is a very difficult statistical problem. There are already several methodologies how to calculate VaR. They all share a common general structure which can be according to Manganelli and Engle (2001) summarized into three points: 1) Mark-to-market[5] the portfolio, 2) Estimate the distribution of portfolio returns, 3) Compute the VaR of the portfolio.

[4]If a risk measure is elicitable, then there exists a scoring function for that risk measure that can be used for comparative tests on models.

[5]Mark-to-market is a method used for measuring the fair value of accounts that can fluctuate over time, such as assets and liabilities. Mark-to-market's goal is to provide a realistic evaluation of given portfolio's current financial situation based on current market conditions.

The major difference between the methodologies lies in approaching the second point. To be more precise the differences are rooted in the estimation of the possible changes in the portfolio's value.

We can classify today's models into three main categories.

- **Parametric** (GARCH and RiskMetrics)
- **Semiparametric** (CAViaR and quasi-maximum likelihood GARCH)
- **Nonparametric** (Historical Simulation)

The results returned by all these methods can differ largely. Beder (1995) showed this fact using eight different methodologies applied on three hypothetical portfolios. As a result, Beder (1995) revealed that some VaR estimates may vary by more than 14 times for the same portfolio. Therefore it is crucial to understand the underlying assumptions and mathematical models used in these models to be able to choose the best fitting model for a given problem.
Since VaR and ES are mostly used with financial data, it is determining to understand them better to be capable to create the best fitting model. The empirical facts about financial data and markets are already well known. The first forays in this direction were works of Mandelbrot (1963) and Fama (1965). Manganelli and Engle (2001) summarize them as follows:

- Financial return distributions are leptokurtotic (they have heavier tails and a higher peak than a normal distribution)
- Equity returns are usually negatively skewed
- Squared returns have significant auto correlation (volatilities of market factors tend to cluster)

Because of the importance of the third characteristic, we can consider market volatilities as quasi-stable, evolving in the long run, but stable in the short period. Most of the VaR models are trying to account for at least some of these empirical characteristics. In this thesis, we are going to limit ourselves to normal GARCH, EGARCH, and CCC-Garch models.

2.3 GARCH model

Generalized autoregressive conditional heteroskedasticity (GARCH) model belongs to the parametric group of models and was developed simultaneously by Bollerslev (1986) and Taylor (1986) in the late eighties. GARCH model is a generalized version of ARCH model firstly proposed by Engle (1982). The motivation for developing GARCH model was that ARCH model, according to Cipra (2008) suffers a couple of drawbacks such as requiring high order and therefore requiring more parameters to estimate and omitting the leverage effect. These defects are overcome by GARCH model. In GARCH model volatility depends not only on lagged squared residual errors as it is in ARCH model but on lagged volatility itself as well.

Definition 10 (GARCH(m, p)). *Let m, p be positive integers. Then given random vector y we de ne GARCH model of orders m, p at time t as*

$$
\begin{aligned}
y_t &= \mu_t + e_t, \\
e_t &= \sigma_t \varepsilon_t, \\
\sigma_t^2 &= \alpha_0 + \sum_{i=1}^{m} \alpha_i e_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2,
\end{aligned}
$$

where μ_t is conditional expected value, ε_t are iid random variables with zero mean and variance equal one[6] *and α and β are parameters which ful ll the following*

$$
\begin{gathered}
\alpha_0 > 0, \;\; \alpha_i \geq 0, \;\; \beta_j \geq 0, \\
\sum_{i=1}^{max[m,p]} (\alpha_i + \beta_i) < 1.
\end{gathered}
$$

Now it is obvious that GARCH$(m, 0)$ actually equals ARCH(m) for any positive integer m. We can as well notice that the equation for the variance of GARCH model is actually ARMA model for squared error series.

It has to be said that GARCH model of orders higher than one is merely used; i.e. the simplest GARCH(1,1) model is the most used model for estimating volatility so far.

[6] There is often assumption for normal or t distribution (Cipra, 2008).

Example 3 (GARCH(1, 1)). *Given random vector y, GARCH model of orders $1, 1$ at time t is defined as*

$$
\begin{aligned}
y_t &= \mu_t + e_t, \\
e_t &= \sigma_t \varepsilon_t, \\
\sigma_t^2 &= \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2,
\end{aligned}
$$

where μ_t is conditional expected value, ε_t are iid random variables with zero mean and variance equal one[7] *and α and β are parameters which fulfill the following*

$$
\begin{gathered}
\alpha_0 > 0,\ \alpha_1 \geq 0,\ \ \beta_1 \geq 0, \\
\alpha_1 + \beta_1 < 1.
\end{gathered}
$$

One of the greatest advantages of GARCH$(1, 1)$ model compared to ARCH model is that equation of variance can be rewritten after substituting lagged variance resulting into

$$
\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
\sigma_t^2 &= \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1(\alpha_0 + \alpha_1 e_{t-2}^2 + \beta_1 \sigma_{t-2}^2) \\
\sigma_t^2 &= \alpha_0 + \alpha_0 \beta_1 + \alpha_1 (e_{t-1}^2 + \beta_1 e_{t-2}^2) + \beta_1^2 \sigma_{t-2}^2 \\
&\ \vdots \\
\sigma_t^2 &= \alpha_0 \frac{1}{1-\beta_1} + \alpha_1 \sum_{i=1}^{\infty} (\beta_1^{i-1} e_{t-i}^2),
\end{aligned}
$$

which is equal to the one used in ARCH(∞) model but using only three parameters instead.

Parameters of these models are most often estimated with the maximum-likelihood method. More specifically a log-likelihood function is formed and the values of parameters that maximise it are sought (Brooks, 2008). However, usually there is a problem with normality assumption, which is violated by financial time series that are generally leptokurtotic. In this case, Quasi-maximum likelihood method for estimating parameters is used instead. Referring to the use of the maximum likelihood method even when the normal distribution assumption is false. When the model is correctly specified, the estimated parameters are consistent and asymptotically normal.

[7] There is often assumption for normal or t distribution (Cipra, 2008).

The standard GARCH model was a great improvement compared to the older ARCH model. However, it still suffers some drawbacks.

- First, the non-negativity conditions for parameters can be violated by the estimated model.
- Second, GARCH model does not take the leverage effect into consideration. It assumes a symmetric response of volatility to positive and negative shocks. This is caused by the square of lagged residuals in the conditional variance equation and this way losing a sign. Assuming a symmetric response of volatility is according to Brooks (2008) in contradiction with empirical knowledge that a negative shock to financial time series is more plausible to cause volatility to increase by more than a positive shock of the same size.

These drawbacks and the fact that analysis of nonlinear time series is a rapidly evolving field led to presenting multiple new models each year. Usually, they are modifications of previous ones and new acronyms such as Beta-Skew-t-EGARCH, EGARCH, etc. are adding up. In this thesis, we will cover EGARCH proposed by Nelson (1991) and CCC-Garch proposed by Bollerslev (1990).

2.3.1 EGARCH

Exponential generalized autoregressive conditional heteroskedasticity (EGARCH) model is the member of an asymmetric branch of models and was proposed by Nelson (1991).

Definition 11 (EGARCH(m, p, q)). *Let m, p, q be positive integers. Then given random vector y we define EGARCH model of orders m, p, q at time t as*

$$y_t = \mu_t + e_t,$$
$$e_t = \sigma_t \varepsilon_t,$$
$$\ln \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \left| \frac{e_{t-i}}{\sigma_{t-i}} \right| + \sum_{j=1}^{p} \beta_j \ln \sigma_{t-j}^2 + \sum_{k=1}^{q} \gamma_k \frac{e_{t-i}}{\sigma_{t-i}},$$

where μ_t is conditional expected value, ε_t are iid random variables with zero mean and variance equal one[8] *and α, β and γ are parameters.*

[8]Originally there was an assumption that ε had Generalised Error Distribution (GED) distribution, which was dropped. Today, almost all applications of EGARCH models utilizes normal distribution (Brooks, 2008).

EGARCH model overcomes the standard GARCH model in several ways. Firstly, we can notice that using $\ln(\sigma_t^2)$ allows us to omit the non-negative conditions for parameters. Secondly, an asymmetric response is accounted for using γ parameters, whenever $\gamma_i \neq 0$. Similarly, as it is with GARCH models, the most used model from EGARCH model family is the simplest one EGARCH(1, 1, 1).

Example 4 (EGARCH(1, 1, 1)). *Given random vector y we de ne EGARCH model of orders 1, 1, 1 at time t as*

$$\begin{aligned} y_t &= \mu_t + e_t, \\ e_t &= \sigma_t \varepsilon_t, \\ \mathit{ln}\ \sigma_t^2 &= \alpha_0 + \alpha_1 \left| \frac{e_{t\ \ 1}}{\sigma_{t\ \ 1}} \right| + \beta_1 \mathit{ln}\ \sigma_{t\ \ 1}^2 + \gamma_1 \frac{e_{t\ \ 1}}{\sigma_{t\ \ 1}}, \end{aligned}$$

where μ_t is conditional expected value, ε_t are iid random variables with zero mean and variance equal one and α, β and γ are parameters.

Cipra (2008) suggests that before using EGARCH model instead of GARCH model it is recommended to test the asymmetry e.g. Engle and NG (1993). To do so, we usually work with residuals $\hat{e}_t$ estimated by standard (symmetric) GARCH model, and using t or F test we test the statistical significance of parameters in the following model.

$$\hat{e}_t^2 = \delta_0 + \delta_1 S_{t\ \ 1} + u_t,$$

where $S_{t\ \ 1} = 1$ for $\hat{e}_t < 0$ and $S_{t\ \ 1} = 0$ otherwise. Additionally, u_t is white noise. If parameter δ_1 in this model is statistically significant it proves that there is asymmetry of volatility in given time series. For more details see Cipra (2008).

2.3.2 Multivariate GARCH and CCC-GARCH

Till now we have mentioned only uni-variate GARCH models, which model volatility or other quantiles based on only one time series and do not consider the interaction between more of them. This is overcome by multivariate GARCH models. The crucial idea of multivariate GARCH models used in finance is that financial markets or even financial assets are tightly connected with each other. This way according to Cipra (2008) the volatility has the tendency to spill over different markets. This correlation plays a crucial role in constructing the optimal portfolios, either we want to bet the money on one card looking for positively correlated assets or hedge the risk with negatively correlated assets. Multivariate GARCH models have an important role in market risk management and they are often used for the calculating capital requirement of banks, or the solvency of insurance companies.

The most simple multivariate GARCH model is two dimensional GARCH$(1,1)$.

Definition 12 (2D GARCH(1, 1)). *Given random vectors y_1, y_2 we define two dimensional GARCH model of orders 1, 1 at time t as*

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_{1,t} \\ \mu_{2,t} \end{pmatrix} + \begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix},$$

$$\begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix} = \begin{pmatrix} \sigma_{1,t}\varepsilon_{1,t} \\ \sigma_{2,t}\varepsilon_{2,t} \end{pmatrix},$$

$$\begin{pmatrix} \sigma^2_{1,t} \\ \sigma^2_{2,t} \end{pmatrix} = \begin{pmatrix} \alpha_{1,0} \\ \alpha_{2,0} \end{pmatrix} + \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} e^2_{1,t-1} \\ e^2_{2,t-1} \end{pmatrix} + \begin{pmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{pmatrix} \begin{pmatrix} \sigma^2_{1,t-1} \\ \sigma^2_{2,t-1} \end{pmatrix},$$

where μ_1, μ_2 are conditional expected values. ε_1, ε_2 are iid random variables with zero mean and variance equal one[9] *and α and β are parameters which fulfill the following*

$$\begin{aligned} \alpha_{1,0}, \alpha_{2,0} &> 0, \\ \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2} &\geq 0, \\ \beta_{1,1}, \beta_{1,2}, \beta_{2,1}, \beta_{2,2} &\geq 0 \\ \alpha_{1,1} + \alpha_{1,2} + \beta_{1,1} + \beta_{1,2} &< 1 \\ \alpha_{2,1} + \alpha_{2,2} + \beta_{2,1} + \beta_{2,2} &< 1 \end{aligned}$$

We can notice that with a growing number of entering time series the number of estimated parameters rises quadratically. This is a major disadvantage of these models. In real life when we want to optimize the portfolio or calculate the capital requirement of a bank we consider a great number of different assets representing different time series. As a simplification of this situation, we use Constant Condition Covariance GARCH or simply CCC-Garch[10] firstly proposed by Bollerslev (1990). This way we get rid of most estimated parameters. We assume that this model returns very similar values to models using dynamic

[9] There is often assumption for normal or t distribution. (Cipra, 2008)

[10] Also known as diagonal GARCH.

conditional correlation, since we assume that correlation between different assets is not that much volatile and mostly sticks to the same value during time. We can write the equation for modeling the volatility using CCC-Garch as follows

$$\sigma_{ij,t} = \alpha_{ij,0} + \alpha_{ij} e_{i,t-1} e_{j,t-1} + \beta_{ij} \sigma_{ij,t-1}.$$

2.4 Optimization of portfolio

Econometric probability models are often fitted to the data with the purpose of optimizing some secondary criterion of interest. In financial econometrics, the best-known optimization problem is dynamic portfolio optimization. The problem lies in initial modelling the risk and returns, predicting the future risk and returns, and finally optimizing the portfolio knowing the predicted future values. There are multiple approaches to this problem, differing mainly in two ways. First, the way the risk is predicted, and the second, the way the risk is defined[11].

We have already covered all of the considered types of risk (2.1 Risk Measures, 2.1.1 Value at Risk, 2.1.2) Expected Shortfall. Now let's define the rate of return or in short the return of the investment as in Cipra (2008).

Definition 13 (Rate of Return). *Given initial value of investment x_s and final value of investment x_f. We define the standard return of the investment as*

$$r_{s,f} = \frac{x_f - x_s}{x_s}.$$

However, in financial econometrics standard returns are used rarely and more common are logarithmic rate of returns or in short logarithmic returns due to their convenient mathematical usage. We define the logarithmic rate of returns as in Cipra (2008).

Definition 14 (Logarithmic Rate of Return). *Given initial value of investment x_s and final value of investment x_f. We define the logarithmic return of the investment as*

$$r_{s,f}^{log} = \log x_f - \log x_s = \log \frac{x_f}{x_s}.$$

We consider optimal portfolio to be the portfolio with minimal risk with regard to given minimal return. The constraint of minimal return allows us to construct

[11] In this thesis, we consider as risk the following Var, VaR and ES

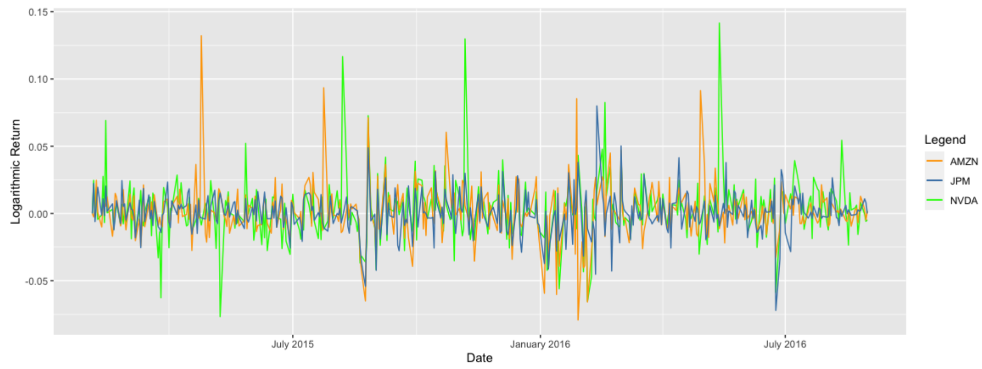

Figure 2.3: An example of a logarithmic returns for three stocks (AMZN, JPM, NVDA). Observed from 01.02.2015 to 31.08.2016

so-called efficient frontier. This approach was introduced in Markowitz (1952). We can define an efficient frontier as in Černý and Holý (2021).

Definition 15 (Efficient Frontier). *We consider e cient frontier to be the set of solutions, ful lling the condition that there does not exist any other solution with the same value of objective function with better passing constraints. Nor any other solution with values of constraints xed, having better value of objective function.*

In our case we understand efficient frontier as the set of portfolios, fulfilling the condition that there does not exist any other portfolio with the same expected return and lower risk. Nor any other portfolio with the same risk and higher expected return, see 2.4. This way multiple solutions are produced. To choose the optimal one, Sharpe ratio of Sharpe (1994) is used, where the solution with the maximum value of Sharpe ratio is chosen.

Definition 16 (Sharpe Ratio). *May μ is expected return of the given portfolio and σ is the risk of the given portfolio. We de ne Sharpe ratio as*

$$S = \frac{\mu}{\sigma}.$$

The idea behind maximizing the Sharpe ratio is that we want high expected portfolio returns, but at the same time, we want low risk.

2.5 Markowitz Model

The theory of portfolio optimization has examined the biggest blossom in the second half of the twentieth century. The biggest breakthrough happened when

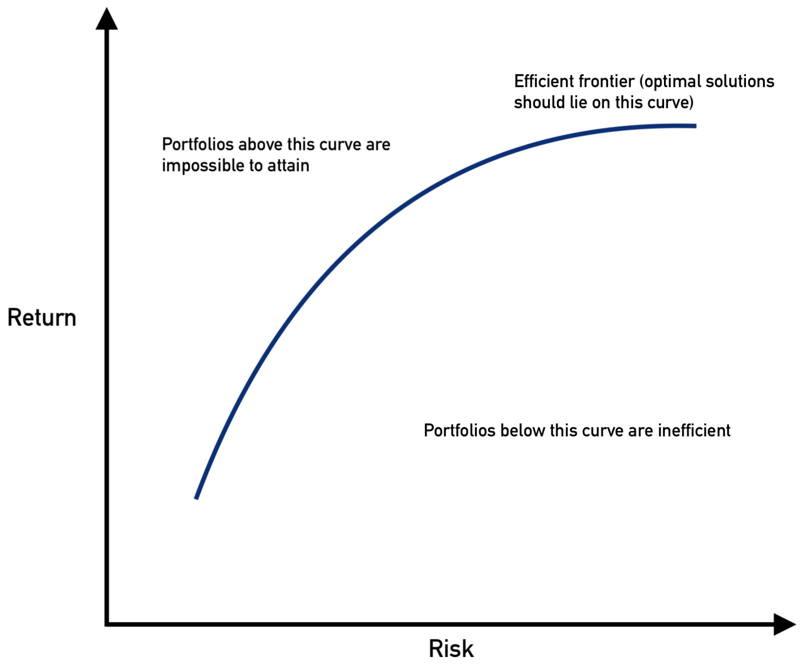

Figure 2.4: Example of efficient frontier.

Harry Markowitz developed his model in Markowitz (1952, 1959), today known as Markowitz or Mean variance model. Markowitz model is today one of the best-known models used for portfolio optimization. This fact is mainly caused by its simplicity. Markowitz model considers the risk of the given portfolio as a variance of its returns. What makes it easily understandable while keeping computational complexity low.

According to Černý and Holý (2021), Markowitz model does not choose assets individually according to their properties but instead considers the properties of the portfolio as a whole as well. What is more Markowitz model considers mutual relationships between given assets. There are more ways to define Markowitz model, but for our use, we define it as follows

Definition 17 (Markowitz model). *May* $n \in \mathbf{N}$ *is number of considered assets,* $w = (w_1, w_2, \cdots, w_n)^T$ *is a vector of their weights in the portfolio,* $y = (y_1, y_2, \cdots, y_n)^T$ *is a vector of their returns and* $\mu = (\mu_1, \mu_2, \cdots, \mu_n)^T = \mu(E(y_1), E(y_2), \cdots, E(y_n))^T$ *is a vector of their expected* *returns. Then we de ne Markowitz model as*

$$\begin{aligned} \min_{w} \; & w^T \Psi w \\ \text{s. t. } \; & w^T \mu \geq \rho, \\ & \sum_{i=1}^{n} w_i = 1, \\ & w_i \geq 0, \text{ for } i = 1, 2, \cdots, n, \end{aligned}$$

where Ψ is a covariance matrix of assets' returns and ρ is minimal required return of the portfolio as a whole. $w_i \geq 0$, for $i = 1, 2, \cdots, n$, tells us that short selling is forbidden.

It is obvious that mutual relationships between assets mentioned before are described with the covariance matrix. The downside of Markowitz model lies in the estimation of this covariance matrix. Markowitz estimates this matrix using the historical method. It is straightforward but on the other hand, it breaks the assumption that volatilities of returns tend to cluster and it lacks dynamic development of volatility. By this means we have a good starting point for portfolio optimization.

2.5.1 Dynamic Markowitz Model

The form of Dynamic Markowitz model is defined identically to the one in Definition 17. The difference lies in the method of estimating the covariance matrix. In Definition 17 Markowitz used the historical method of estimating the covariance matrix in his model. On the other hand, in Dynamic Markowitz model we propose a slightly more sophisticated way, inspired by Blasques (2017). For estimating the covariance matrix used in the model we use CCC-Garch method proposed by Bollerslev (1990). We assume that the covariance matrix is constant over time, however its diagonal with conditional variances is not. This approach lies in calculating the covariance matrix historically but after that replacing its diagonal with conditional variances obtained using GARCH or EGARCH methodologies or in third case with 95 % expected shortfall. This way we receive a dynamic model which takes into mind assumptions, which were broken by the previous approach.

2.6 Mean-ES

Another approach to portfolio optimization, which is more advanced than the previously mentioned Markowitz Model is Mean-ES model. Mean-ES is based on minimization of portfolio's risk expressed by ES and this way minimizing the risk of high losses. This model has its roots in Mean-VaR model but because of using ES instead of VaR is more consistent when considering not normally distributed loss functions (Krokhmal et al., 2003).

However, optimization of ES on its own is a difficult and computationally intensive task because of the integral used in the definition of ES. Therefore, the conveniently solvable definition of Mean-ES model was proposed in Rockafellar and Uryasev (2000). Their approach avoids using integral and uses only its approximation. Next, it calculates VaR and optimizes ES simultaneously.

The function that is being minimized is

$$F_\alpha(w,\beta) = \beta + \alpha^{-1} \int_{y \in \mathbf{R}^n} [f(w,y) - \beta]^+ p(y) d(y),$$

where w represents weights of the assets in the portfolio, y represents returns of the assets in the portfolio, α is confidence level, β is $\text{VaR}_\alpha(w)$, $p(y)$ is the probability density of y and $f(w,y)$ is a function representing portfolio loss. Additionally

$$[t]^+ = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

In Rockafellar and Uryasev (2000) it is shown that

$$\text{ES}_\alpha(w) = \min_{\beta \in \mathbf{R}} F_\alpha(w,\beta)$$

$$\min_{w \in \langle 0;1 \rangle} \text{ES}_\alpha(w) = \min_{(w,\beta) \in \langle 0;1 \rangle \times \mathbf{R}} F_\alpha(w,\beta).$$

In this case, we do not assume any distribution of the random variable y. Therefore, it has to be approximated from the previous observations. Having known q observations of random vector $y = (y_1, y_2, \cdots, y_q)$, $F_\alpha(w,\beta)$ can be approximated as follows

$$\hat{F}_\alpha(w,\beta) = \beta + \frac{1}{q\alpha} \sum_{k=1}^{q} [f(w,y_k) - \beta]^+.$$

This way we got everything needed for Mean-ES model definiton.

Definition 18 (Mean ES). *May* $n \in \mathbf{N}$ *is number of considered assets,* $w = (w_1, w_2, \cdots, w_n)^T$ *is a vector of their weights in the portfolio,* $y = (y_1, y_2, \cdots, y_n)^T$ *is a vector of their returns,* $\mu = (\mu_1, \mu_2, \cdots, \mu_n)^T = \mu(E(y_1), E(y_2), \cdots, E(y_n))^T$ *is a vector of their expected returns,* $q \in \mathbf{N}$ *is a number of observations of* y *we have,* $\alpha \in [0; 1]$ *is a con dence level and* ρ *is a minimal required return of the portfolio as a whole. Then we de ne Mean-ES model as*

$$\begin{aligned} \min_{(w,\beta) 2 <0;1> \quad \mathbf{R}} \quad & \beta + \frac{1}{q\alpha} \sum_{k=1}^{q} [w^T y_k - \beta]^+ \\ \text{s. t.} \quad & w^T \mu \geq \rho, \\ & \sum_{i=1}^{n} w_i = 1, \\ & w_i \geq 0, \ \text{for } i = 1, 2, \cdots, n. \end{aligned}$$

3. Data Description

Data used in the empirical part of this thesis are based on stocks of companies, all listed in S&P 500[12] index. This index consists of 500 of the largest publicly-traded companies in the U.S. in different fields such as information technology, finance, health care, etc... To keep the diversity in our data-set we chose the well-known representatives for each field.

Namely they are Adobe Inc. (ADBE), Amazon.com Inc. (AMZN), Boeing Company (BA), Boston Scientific (BS), Caterpillar Inc. (CAT), CVS Health (CVS), Chevron Corp. (CVX), DuPont de Nemours Inc. (DD), The Walt Disney Company (DIS), Equity Residential (ER), Facebook Inc. (FB), Goldman Sachs Group (GS), JPMorgan Chase Co. (JPM), Coca-Cola Company (KO), Newmont Corporation (NEM), Nvidia Corporation (NVDA), Oracle Corp. (ORCL), Philip Morris International (PM), Starbucks Corp. (SBUX), Walgreens Boots Alliance Inc. (WBA) and Xcel Energy Inc. (XEL).

We gathered the closing prices for each company on New York Stock Exchange from February 1, 2015, to February 1, 2020, and from February 1, 2020, to June 30, 2020, dividing this way the whole data set into two parts the learning part and the testing part. The reason why the data-set was divided exactly this way is caused by the fact that at the very beginning of 2020 the global Covid pandemic has started and negatively affected the financial markets. The goal of this thesis is to show if using advanced methods in the estimation of market risk is satisfactory in times of financial recession.

3.1 Data preparation

As an example of our approach we chose three representatives Amazon.com Inc., JPMorgan Chase Co. and Nvidia Corporation, with tick names as follows AMZN, JPM and NVDA. First of all we need the closing prices of these stock on New York Stock Exchange from February 1, 2015, to February 1, 2020. We can observe their change in time on Figure 3.1. We can say the Amazon.com Inc.'s stock price was developing the most, and on the other hand we can see that

[2]The S&P 500 Index stands for the Standard & Poor's 500 Index and it is a market-capitalization-weighted index of 500 of the largest publicly traded companies in the U.S.

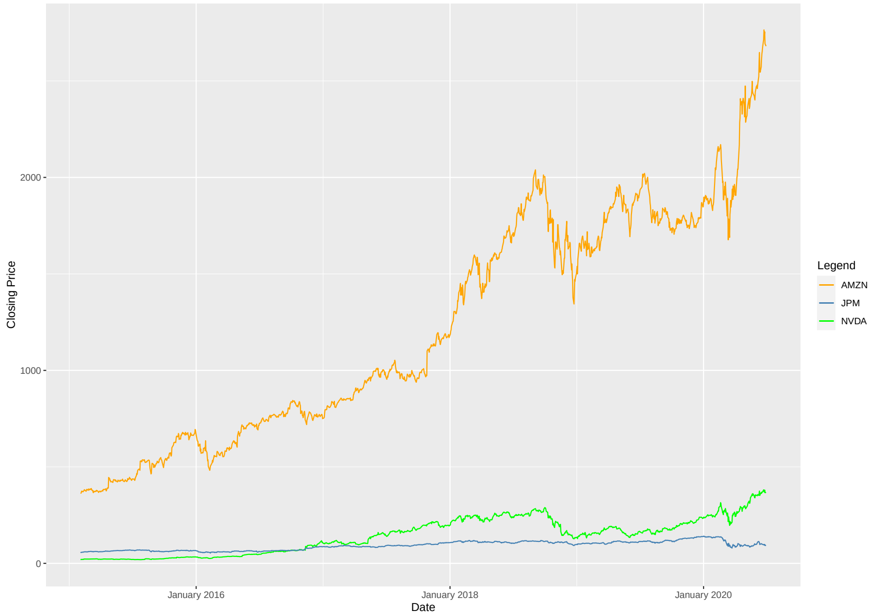

Figure 3.1: An example of closing prices on NYSE for three stocks (AMZN, JPM, NVDA). Observed from February 1, 2015, to June 30, 2020

JPMorgan Chase Co.'s stock price was pretty stable. Next, we are going to use logarithmic returns. They are obtained by using Definition 14. We can see them in Figure 3.2. It can be observed that the assumption of volatility clustering is empirically proven to be right. In Figure 3.3 it can be seen that the distribution of their logarithmic returns over time. It can be clearly seen that the distributions have higher kurtosis than Standard Normal distribution often considered when working with financial data and as well heavier tails. This can be seen on NVIDIA Corporation the best. The data can be considered to be prepared for the next analysis. With the same practice, we prepare all of the other time-series in the learning data-set (containing data from February 1, 2015, to February 1, 2020) and exactly the same practice we apply on the training data-set (containing data from February 1, 2020, to June 30, 2020).

Next, we examine the descriptive statistics of logarithmic returns of our stocks. First of all, we examine the training data-set. The descriptive statistics are summarized in Table 3.1. This data set's time series consists of 1258 observations each. We can see that their mean is almost zero or slightly above zero. This is in accord with the nature of the firm, which always wants to stay in profit and develop itself. The highest mean 0.00201 has NVDA and on the other hand, the

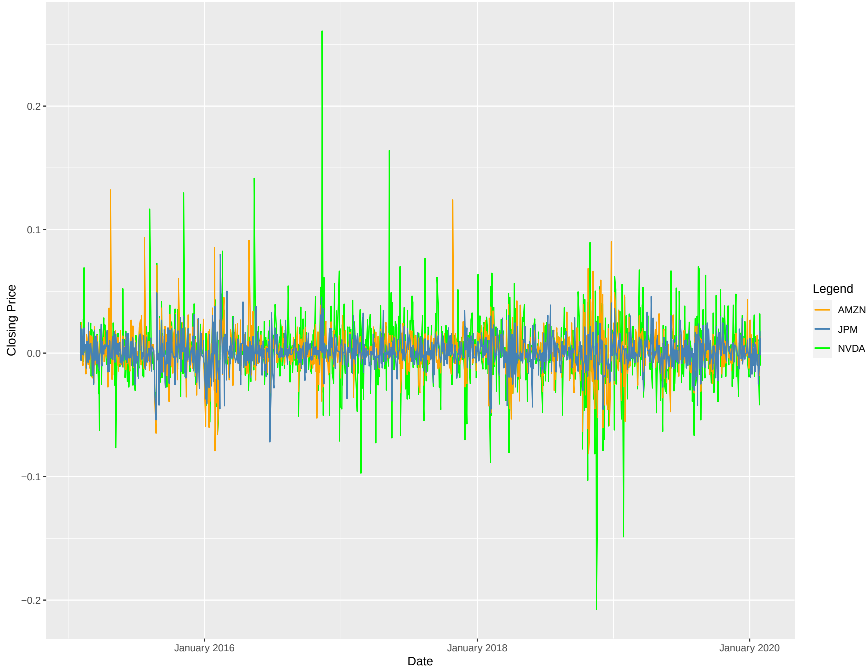

Figure 3.2: An example of logarithmic returns of three stocks (AMZN, JPM, NVDA). Observed from February 1, 2015, to February 1, 2020

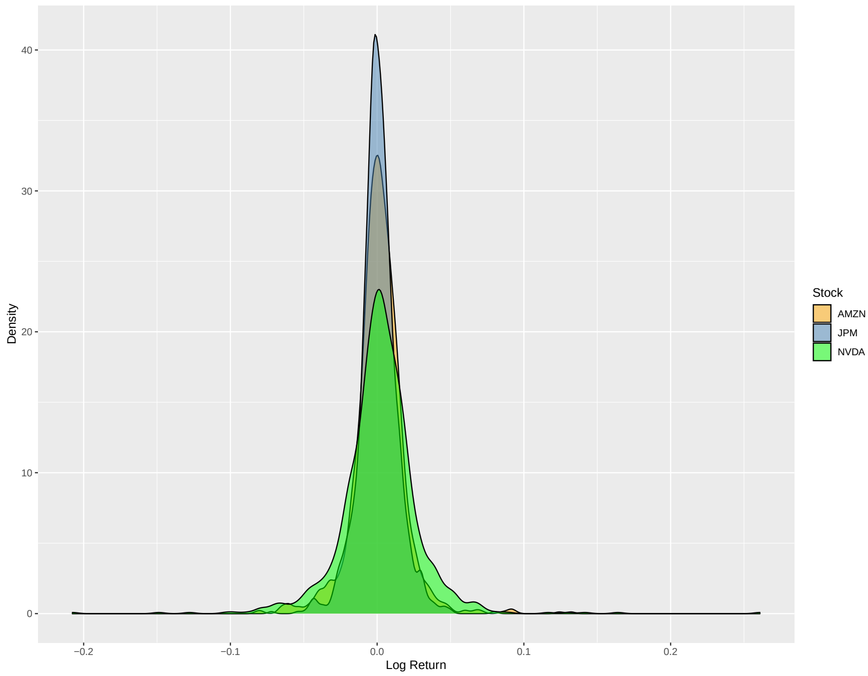

Figure 3.3: An example of distribution of the logarithmic returns of three stocks (AMZN, JPM, NVDA). Observed from February 1, 2015, to February 1, 2020

lowest one -0.00029 has WBA. Next, we can notice that their standard deviation is almost the same. The highest standard deviation 0.027 has NVDA and the lowest one 0.009 has KA. The lowest logarithmic return, or in other words the biggest loss, in the whole data-set occurred to FB and it was -0.210. On the opposite side, the highest logarithmic return observed happened to NVDA and it was 0.261. We can notice that the majority of the stocks' distributions are negatively skewed what indicates that their distributions are not symmetrical and generally stocks experience higher losses than profits. And the last point serves as an example of why the normality assumption for financial data is not quite correct. We can see that almost every stock's distribution (except CAT, CVX, GS, XEL) has high kurtosis ranging from 3.295 to 24.051 making these distributions leptokurtic.

Secondly, we look at the descriptive statistics of the testing data set. They are summarized in Table 3.2. This data set's time series consists of 103 observations each. And the time, during which they were observed, reflects the global pandemic of Covid 19 which resulted in a finance market recession. We can see that their means are still almost zero but almost all of them are all of a sudden negative. This is caused exactly by the finance market recession. The highest mean 0.00414 has still NVDA but it is followed by AMZN with 0.00282, which reflects the situation that since people had to stay at home because of the quarantine they started to shop online more. On the other hand, the lowest one -0.000471 has BA. This is understandable because the majority of flights have been cancelled and airlines are not buying new airplanes because of this situation. Next, we can notice that their standard deviation among the stocks is almost the same but multiple times higher than it was. The highest standard deviation 0.079 has BA, because of the tremendous fall on NYSE. The lowest one 0.026 has AMZN. This way we can conclude that in terms of higher risk, the global pandemic negatively affected all of the observed companies. The lowest logarithmic return, or in other words the biggest loss, in the whole data set occurred to BA and it was -0.272. On the opposite side, the highest logarithmic return observed happened to BA as well and it was 0.218. This was caused by slacking the restrictions later on. It can be noticed that the majority of the stocks' distributions are negatively skewed again what indicates that companies experienced mostly losses during this time. Finally, because we have only 103 observations the kurtosis is not so high as in the previous case but still is higher than the normal distribution's kurtosis.

Stock	Mean	St. Dev.	Min	Max	Skew.	Kurt.
ADBE	0.00128	0.016	-0.087	0.115	-0.111	5.980
AMZN	0.00130	0.018	-0.081	0.132	0.522	7.657
BA	0.00063	0.016	-0.094	0.094	-0.320	3.814
BSX	0.00087	0.016	-0.077	0.117	0.484	7.111
CAT	0.00041	0.017	-0.096	0.076	-0.266	2.924
CVS	-0.00028	0.015	-0.126	0.072	-0.767	6.326
CVX	0.00004	0.014	-0.057	0.061	0.000	2.103
DD	-0.00016	0.017	-0.097	0.164	0.700	11.039
DIS	0.00032	0.013	-0.096	0.109	0.119	10.289
EQR	0.00007	0.011	-0.082	0.036	-0.813	3.714
FB	0.00082	0.018	-0.210	0.144	-1.005	20.572
GS	0.00026	0.015	-0.077	0.091	-0.286	2.749
JPM	0.00071	0.013	-0.072	0.080	-0.043	3.295
KO	0.00028	0.009	-0.088	0.059	-0.895	10.003
NEM	0.00046	0.022	-0.130	0.107	-0.358	3.869
NVDA	0.00201	0.027	-0.208	0.261	0.337	13.720
ORCLA	0.00018	0.013	-0.099	0.082	-0.642	7.849
PM	0.00003	0.013	-0.169	0.084	-1.740	24.051
SBUX	0.00053	0.013	-0.097	0.093	-0.429	9.555
WBA	-0.00029	0.016	-0.137	0.062	-1.146	9.261
XEL	0.00048	0.010	-0.051	0.031	-0.639	1.859

Table 3.1: Descriptive statistics of each stock's logarithmic returns from February 1, 2015, to February 1, 2020 (Training data-set)

Stock	Mean	St. Dev.	Min	Max	Skew.	Kurt.
ADBE	0.00165	0.040	-0.160	0.163	-0.146	4.255
AMZN	0.00282	0.026	-0.083	0.068	-0.434	1.160
BA	-0.00471	0.079	-0.272	0.218	-0.211	1.642
BSX	-0.00192	0.038	-0.160	0.088	-0.815	1.959
CAT	-0.00034	0.039	-0.154	0.098	-0.651	1.720
CVS	-0.00037	0.033	-0.131	0.103	-0.388	3.489
CVX	-0.00187	0.055	-0.250	0.205	-0.883	5.916
DD	0.00012	0.044	-0.142	0.132	-0.212	1.169
DIS	-0.00230	0.040	-0.139	0.135	-0.077	1.926
EQR	-0.00352	0.042	-0.185	0.102	-0.513	2.493
FB	0.00075	0.036	-0.154	0.097	-0.731	3.188
GS	-0.00205	0.047	-0.136	0.162	-0.046	1.955
JPM	-0.00350	0.048	-0.162	0.166	-0.024	1.926
KO	-0.00270	0.030	-0.102	0.063	-0.457	0.820
NEM	0.00259	0.038	-0.118	0.131	0.254	2.703
NVDA	0.00414	0.048	-0.204	0.158	-0.589	3.158
ORCLA	0.00030	0.037	-0.117	0.186	0.807	6.897
PM	-0.00158	0.036	-0.133	0.096	-0.893	2.912
SBUX	-0.00153	0.041	-0.177	0.137	-0.240	3.621
WBA	-0.00180	0.039	-0.116	0.119	0.035	0.902
XEL	-0.00103	0.036	-0.136	0.107	-0.536	3.242

Table 3.2: Describing statistics of each stock's logarithmic returns from February 1, 2020, to June 30, 2020 (Testing data-set)

It can be seen that every stock's distribution has much lower kurtosis, ranging from 0.820 to 6.897, than it was in the training data-set case. This is caused by the limited number of observations with multiple times higher standard deviation than in the training data set.

Another important descriptive statistic when talking about optimizing a portfolio is the correlation between each asset as was already mentioned before. Firstly, we can look at Figure 3.4, where we can see the correlation plot between each company's logarithmic returns from February 1, 2015, to February 1, 2020. And see that each two of the companies are positively correlated. This can be caused interconnection of the markets. We can notice that the least correlated company is NEM, which is DuPont de Nemours Inc... And two the most correlated are JPM and GS, which are JPMorgan Chase and Co. and Goldman Sachs Group, it's pretty understandable since both of them are one of the greatest investment banks operating in the same field. Nevertheless, this correlation plot says us that when one company prospers or losses it affects the other companies listed here in a similar way.

Secondly, we can look at Figure 3.5, where we can see the correlation plot between each company's logarithmic returns from February 1, 2020, to June 30, 2020. And see that each two of the companies are even more positively correlated. This is caused by the global economic recession which affects almost all fields. We can notice that the least correlated company is still NEM, which is DuPont de Nemours Inc.. And two the most correlated are JPM and GS, which are JPMorgan Chase and Co. and Goldman Sachs Group, but now even more. Nevertheless, this correlation plot says that during the global economic recession are companies correlated even more. This high correlation between each company's logarithmic returns makes optimization and diversification of the portfolio even more challenging.

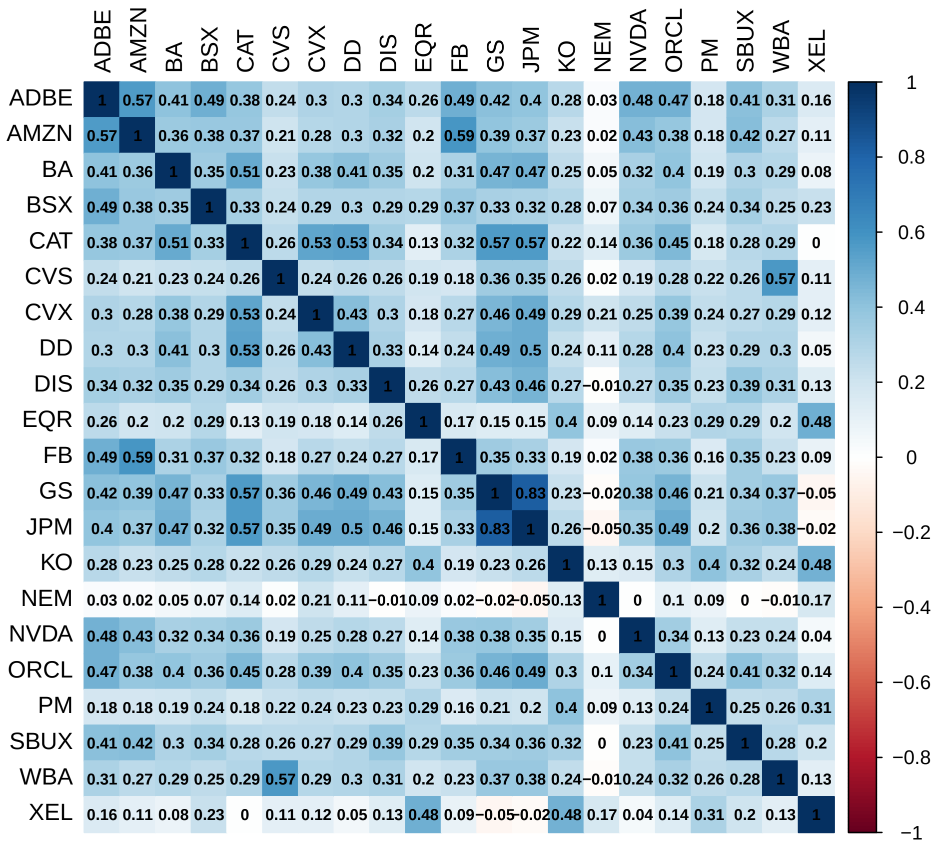

Figure 3.4: Correlation between logarithmic returns of our stocks. Observed from February 1, 2015, to February 1, 2020

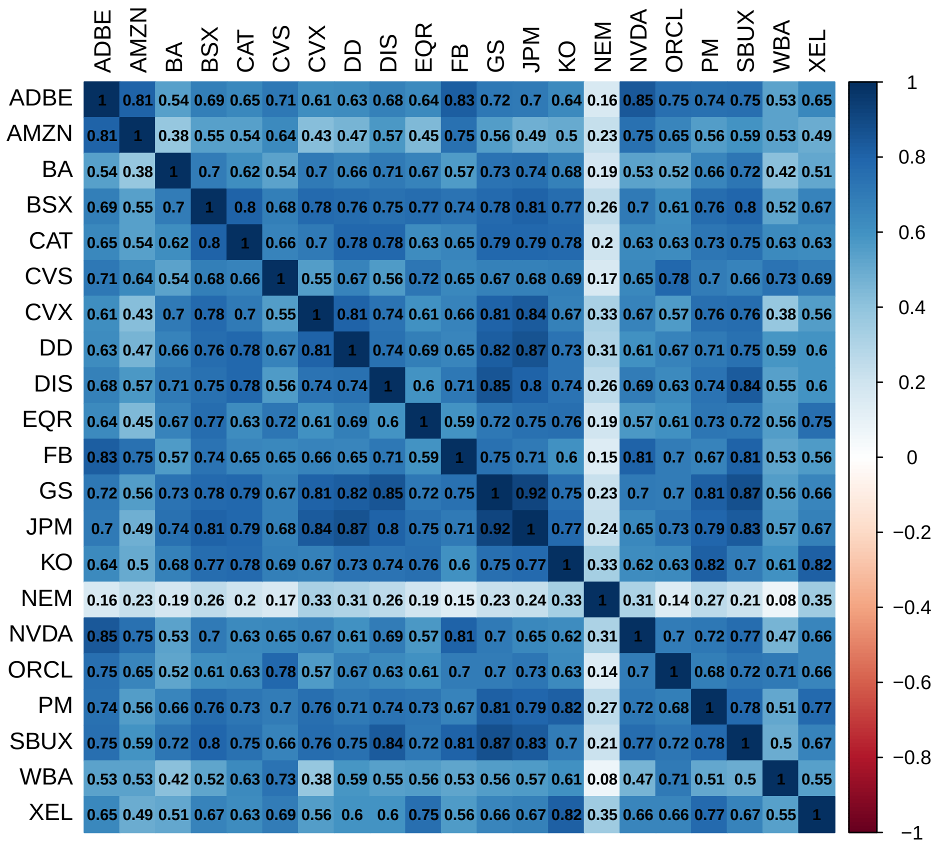

Figure 3.5: Correlation between logarithmic returns of our stocks. Observed from February 1, 2020, to June 30, 2020

4. Empirical Analysis

In this part of the thesis we, first of all, forecast the volatility of our stocks, using GARCH(1,1) and EGARCH(1,1,1) models. Then we use these results in our portfolio optimization using Markowitz model. Secondly, we repeat this process, but instead of forecasting the volatility of our stocks, we forecast 95% ES of our stocks. Finally, we use Mean-ES model. The optimization models are applied 103 times, which equals NYSE functioning days from February 1, 2020, to June 30, 2020. This way we can observe changes in the weights of each stock in our portfolio and compare expected losses and profit between models used, and this way decide which approach suits the financial recession the most.

4.1 Predicting the volatility using GARCH(1,1) and EGARCH(1,1,1)

At the very beginning, we are going to forecast volatility one step ahead using GARCH(1,1) model. We use the training data-set to model the very first model for each stock's logarithmic returns. In the next step, we forecast the volatility one step ahead a save its value for further optimization. Next, we extend the training data set by a new row, which is in this case, the first row in the testing data set and it corresponds to the next days' logarithmic returns. We reestimate the model using GARCH(1,1) based on extended data set and afterwards we extend the given data set with the next row from the testing data set, corresponding to the next days' logarithmic returns. We repeat these steps till we are not at the end of our testing data set. This way we get the development in forecasted volatility over time (February 1, 2020, to June 30, 2020).

The very same approach we use in the next step, where we are modelling the volatility using EGARCH(1,1,1), assuming Student's t distribution. This model overcomes the GARCH(1,1) in its shortcomings and should provide us more precise results.

The comparison of both models can be seen in Figures 4.1 and 4.2, where we chose AMZN stocks, which encountered the least losses during the examined time, and BA stocks, which on the other hand encountered the highest losses during the examined time. The most interesting observation is that EGARCH

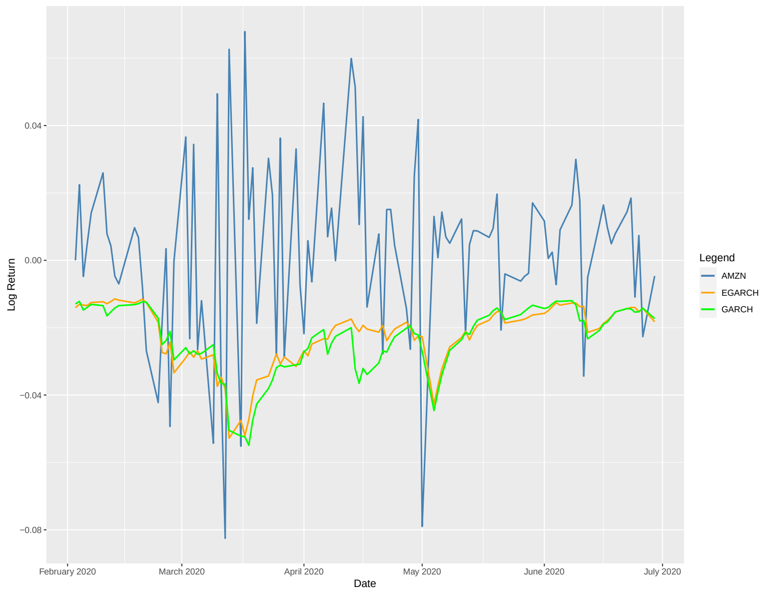

Figure 4.1: An example of volatility change of logarithmic returns of AMZN stocks. Model by GARCH(1,1) and EGARCH(1,1,1) models. Observed from February 1, 2020, to Junbe 30, 2020

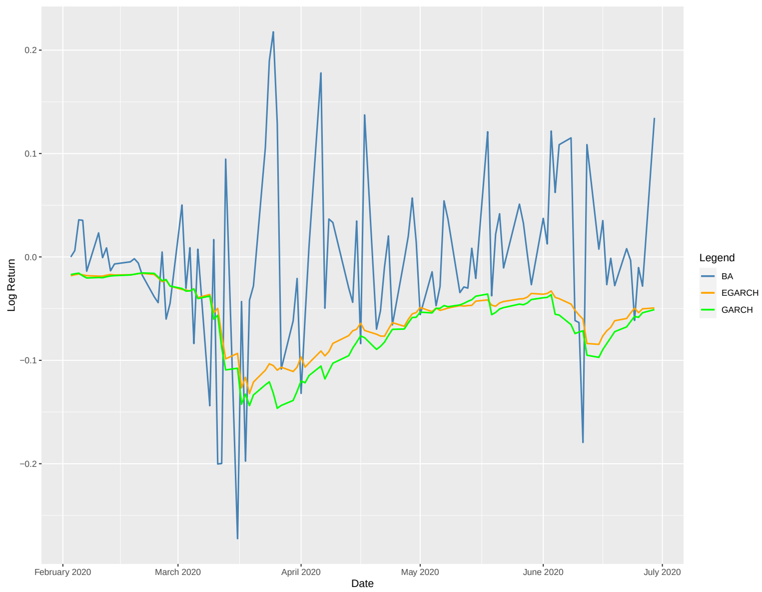

Figure 4.2: An example of volatility change of logarithmic returns of BA stocks. Model by GARCH(1,1) and EGARCH(1,1,1) models. Observed from February 1, 2020, to June 30, 2020

model is clearly not affected by positive fluctuation. This is the best seen from April to May since this time horizon reflects the time just after the so-called first wave and during this time multiple restrictions were released. In these graphs, the volatility is plotted with a negative sign to better reflect the risk of a loss.

4.2 Optimizing of the portfolio using Mean Var GARCH(1,1)

In this section, we optimize the portfolio using Markowitz model, where we are using the GARCH(1,1) model for forecasting the volatility. To do so we use MATLAB's built-in function *quadprog* which solves quadratic programming problems. At first, we import the learning data consisting of time series of considered stocks log returns and calculate their covariances. Afterwards, we use GARCH(1,1) model for each stock to predict the future development of their volatility. And replace the diagonal of calculated covariances with predicted variances for each stock. This way we implement the CCC-Garch, where we assume that the volatility of each stock and their covariances are dynamic, but stable during the day. Afterwards, the minimal and maximal expected log return is found and we estimate Markowitz model for each log return constraint of hundred steps in between this gap. The expected log return is calculated with Exponentially weighted moving average (EWMA) method applied on the latest one hundred observations and using $\lambda = 0.94$, as it is often used by RiskMetrics™ [12]. This way we get the effective frontier, see Figure 4.3. We can see its concave plot ending at the point with the maximum possible return, receivable by investing all the resources in the stock with the highest expected return. Afterwards, we find the highest Sharpe Ratio, again see Figure 4.3. The change in the weights can be seen in Figure 4.4, supporting the previous statement. It is clear that at first, the portfolio is more diverse minimizing the risk, but it becomes less diverse therefore riskier with a higher expected return.

Next, we add a new observation from the training data set to the learning data set and repeating the previous steps. This way we get portfolio optimization for each day from February 1, 2020, to June 30, 2020, when NYSE was opened.

[2] RiskMetrics™ is a financial risk management company

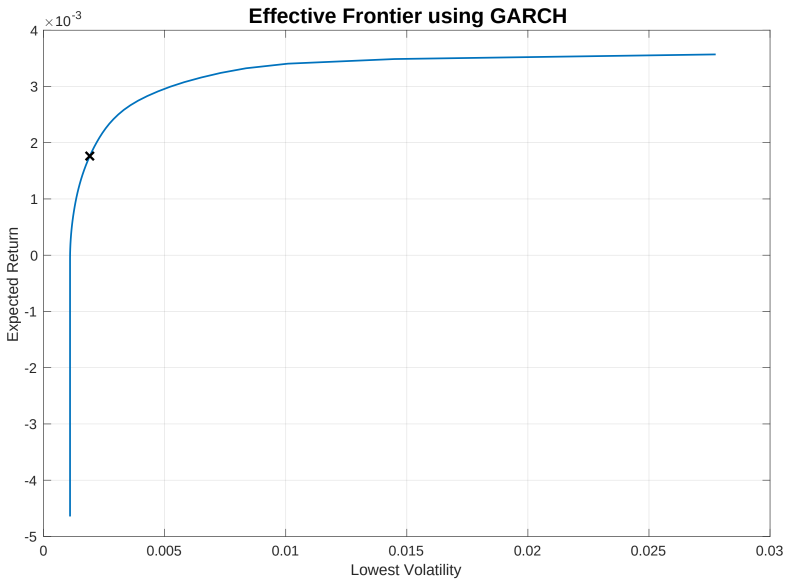

Figure 4.3: Effective Frontier - February 3, 2020.
Where black cross represents the portfolio with the maximum Sharpe Ratio.

In Figure 4.5 we can see the change in volatility over time. We can notice the first peak around the 30th day, caused by the initial pandemic spread. During the following days, it slowly returns to the previous state but in the middle of June 2020, it peaks again. This time it is caused by the global pandemic spread.

Next, we can see Figure 4.6, which shows the change of estimated Sharpe Ratio over time. Around the 30th, day we can observe a massive drop caused by an initial peak in the volatility. Later on, we can observe that Sharpe Ratio slowly returns to the previous state, which is caused at first by declining volatility and later by inclining log returns.

Finally, we can look at Figure 4.7, where we can see the change of weights of portfolio assets over time. Two all market shifts can be observed. The first one in the middle of February 2020, as a reaction to WHO, which declared the COVID-19 outbreak, and the second one in the middle of March 2020, as a reaction to the countries closing their economies. These shifts were expected and support the functioning of this optimization approach. We can observe that during the whole duration our portfolio consists of almost all stocks, different only in weights. We

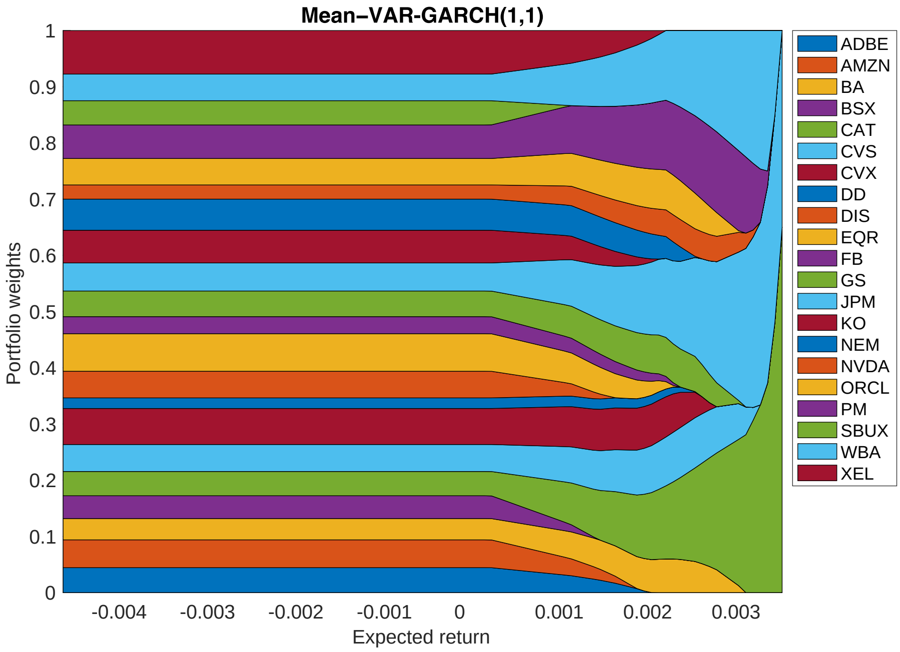

Figure 4.4: Weights of portfolio assets on the Effective Frontier

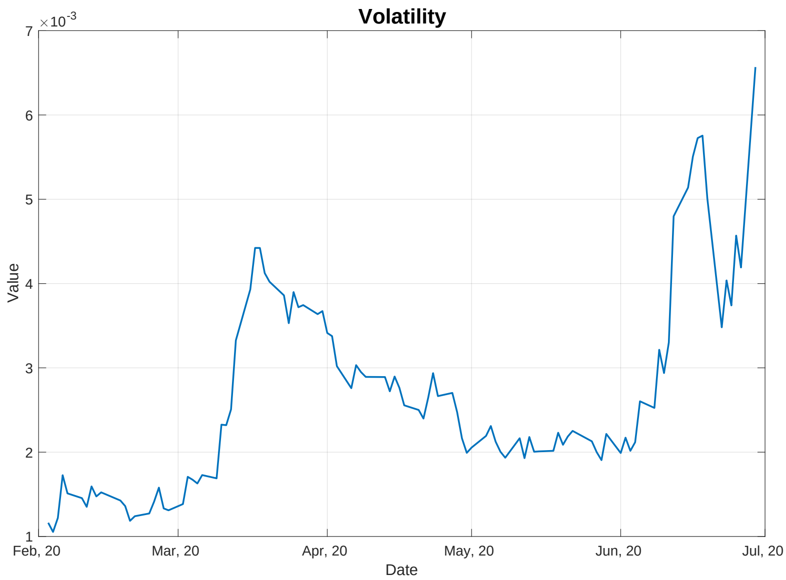

Figure 4.5: Estimated volatility of the portfolio over the time

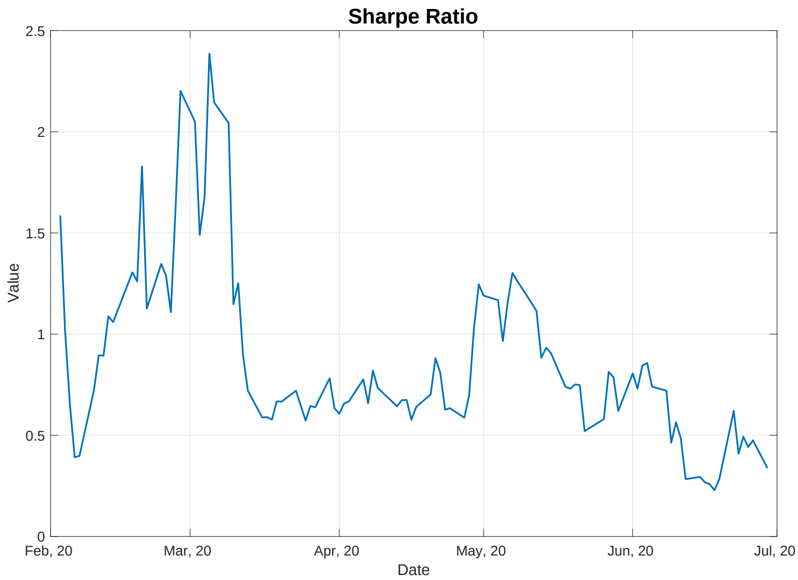

Figure 4.6: Sharpe Ratio over the time.

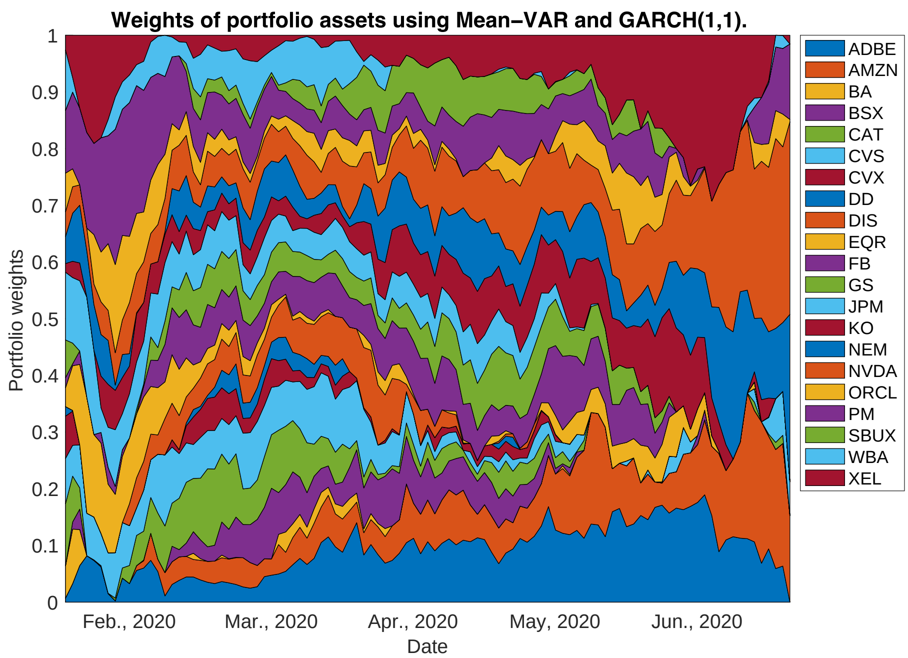

Figure 4.7: Weights of portfolio assets over the time.

STOCK	Feb. 3, 2020	Mar. 9, 2020	Mar. 16, 2020	Jun. 24, 2020
ADBE	0.008	0.041	0.073	0.088
AMZN	0.002	0.050	0.041	0.199
BA	0.055	0.001	0.026	0.000
BSX	0.000	0.065	0.062	0.000
CAT	0.100	0.091	0.085	0.008
CVS	0.078	0.066	0.091	0.037
CVX	0.066	0.050	0.027	0.000
DD	0.031	0.045	0.035	0.000
DIS	0.000	0.052	0.072	0.000
EQR	0.036	0.022	0.013	0.003
FB	0.025	0.056	0.065	0.000
GS	0.071	0.070	0.050	0.000
JPM	0.127	0.082	0.047	0.000
KO	0.019	0.036	0.022	0.035
NEM	0.043	0.020	0.029	0.155
NVDA	0.052	0.050	0.057	0.252
ORCL	0.077	0.032	0.040	0.053
PM	0.103	0.065	0.046	0.106
SBUX	0.000	0.024	0.037	0.000
WBA	0.084	0.058	0.080	0.008
XEL	0.022	0.022	0.003	0.058

Table 4.1: Weights of portfolio assets for specific days.

can notice that the BA stocks are almost missing from our portfolio, as expected since almost all flights were cancelled and airlines stopped ordering new planes. Especially, this can be noticed in Table 4.1, when the weight of BA stocks dropped from 0.055 on February 3, 2020, to 0.001 on March 9, 2020. On the other hand, we can observe the growth of the AMZN weight, since people stayed at home and started using e-shops even more. We can also notice that NVDA, which had the highest mean return during that time since the increase of net revenue of the company, has the second-biggest share of our portfolio. The table with estimated weights of assets in the portfolio can be seen in Table 4.1, the presented days were chosen because around 30th the big shift on financial markets.

To sum up, this approach brought us gross depreciation of 2.15%. We got this number by transforming log returns to standard returns by exponentiation them. Afterwards, multiplying the returns with each other and given weights and finally subtracting the 1 and multiplying by 100 to get the percentage. This result is satisfactory since the loss was not that big but still our portfolio was not profitable. What encourages us to look for other optimization strategies.

4.3 Optimizing of the portfolio using Mean Var EGARCH(1,1,1)

In this section we optimize the portfolio using Markowitz model, where we are using EGARCH(1, 1, 1) model for forecasting the volatility instead of GARCH(1, 1) as it was in the previous case. This way we expect more profitable results since EGARCH(1, 1, 1) accounts for asymmetric response and therefore is suited more for financial data like these. To do so we again use MATLAB's built-in function *quadprog* which solves quadratic programming problems. The first step is the same as it was in the previous case. We are importing the learning data consisting of time series of considered stocks log returns, and calculate their covariances. Next comes the difference, where instead of using GARCH(1,1) for each stock to determine the future development of their volatility, we use EGARCH(1,1,1) model. Afterwards, we replace the diagonal of calculated covariances with predicted variances for each stock. The minimal and maximal expected log return, using EWMA method with $\lambda = 0.94$, is found and we estimate the Markowitz model for each log return constraint of hundred steps in between this gap. This way we get the effective frontier, see Figure 4.8. We can see its concave plot ending at the point with the maximum possible return, receivable by investing all the resources in the stock with the highest expected return. The highest Sharpe Ratio is chosen, again see Figure 4.8. Compared with 4.3, it can be noticed that the maximal volatility now is slightly lower, but the shape of the effective frontier is almost identical. The change in the weights can be seen in Figure 4.9. It is almost identical to the change in weights in Figure 4.4.

Following this, we add a new observation, one by one, from the training data set to the learning data set and repeating the previous steps. This way we get portfolio optimization for each day from February 1, 2020, to June 30, 2020, when

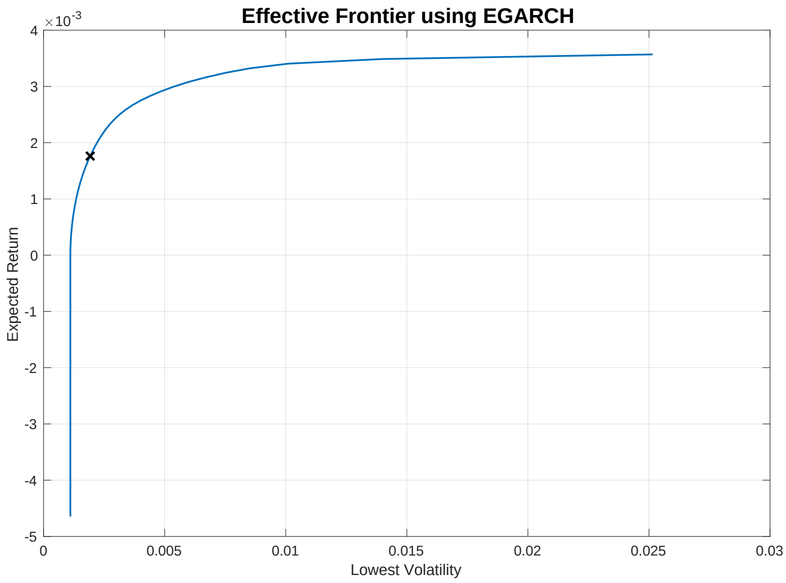

Figure 4.8: Effective Frontier - February 3, 2020
Where black cross represents the portfolio with the maximum Sharpe Ratio.

NYSE was opened. Several plots are provided.

Figure 4.10 shows the change in volatility over time. The first peak can be noticed around March 20, 2020. However, compared with Figure 4.5 is not that high. During the following days, it slowly declines with the next high peak coming in the middle of June 2020, this time higher than it is in Figure 4.5.

In Figure 4.11, we can see how the estimated Sharpe Ratio changed over time. Around March 20, 2020, massive drop caused by the first peak in the volatility can be observed. Later on, it can be observed that Sharpe Ratio slowly returns to the previous state or even declines, which is caused by declining volatility at first and later on by inclining log returns.

Finally, let's take a look at Figure 4.12, where the change of weights of portfolio assets over time can be seen. It can be noticed that behaviour of our portfolio weights of assets is almost identical to the previous one mentioned in Section 4.2. Therefore, we expect the total profit to be very similar to that in Section 4.2. The table with estimated weights of assets in the portfolio can be seen in Table

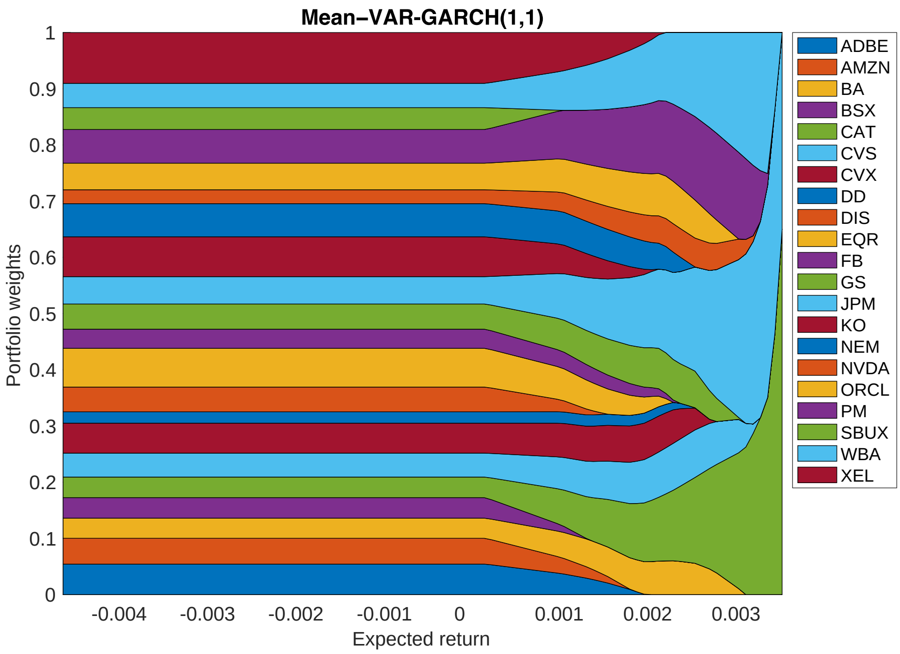

Figure 4.9: Weights of portfolio assets on the Effective Frontier

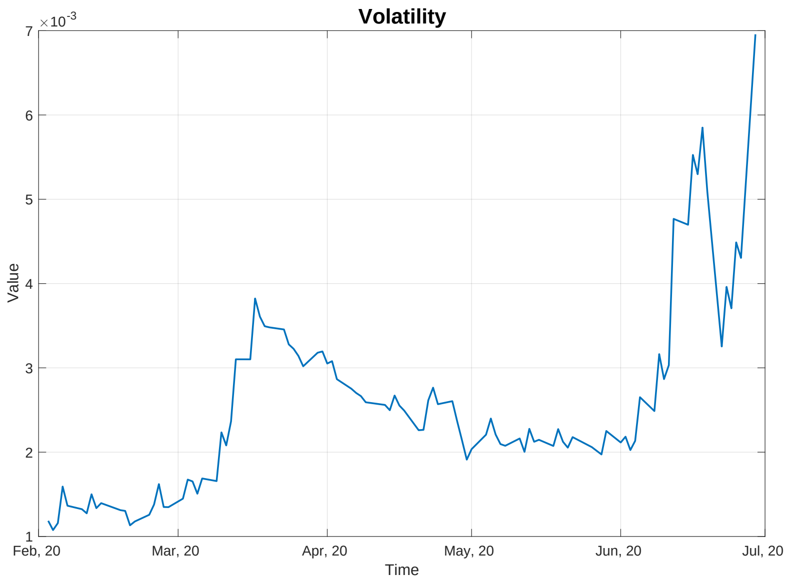

Figure 4.10: Volatility of the portfolio over the time

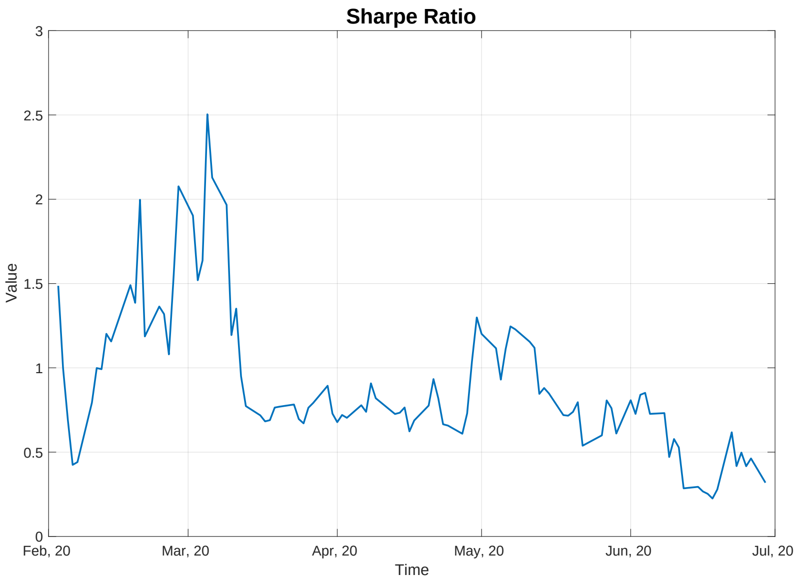

Figure 4.11: Sharpe Ratio over the time.

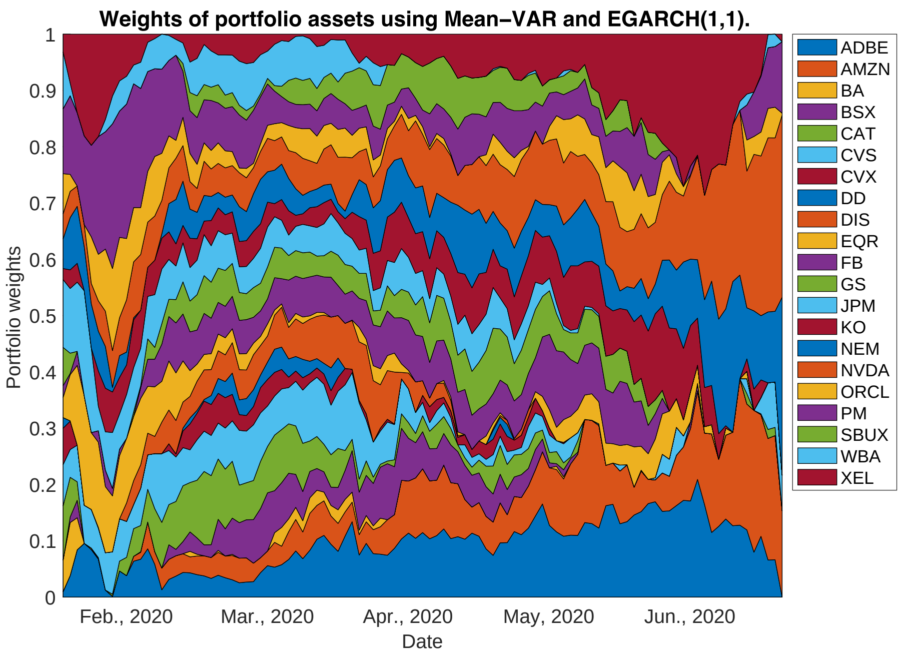

Figure 4.12: Weights of portfolio assets over the time.

STOCK	Feb. 3, 2020	Mar. 9, 2020	Mar. 16, 2020	Jun. 24, 2020
ADBE	0.009	0.031	0.068	0.108
AMZN	0.001	0.043	0.047	0.216
BA	0.056	0.002	0.020	0.000
BSX	0.000	0.064	0.052	0.000
CAT	0.096	0.101	0.086	0.001
CVS	0.073	0.072	0.095	0.029
CVX	0.065	0.047	0.022	0.000
DD	0.019	0.038	0.033	0.000
DIS	0.000	0.054	0.064	0.000
EQR	0.036	0.025	0.012	0.000
FB	0.021	0.058	0.068	0.000
GS	0.068	0.058	0.046	0.000
JPM	0.120	0.054	0.050	0.000
KO	0.020	0.038	0.019	0.031
NEM	0.053	0.030	0.033	0.114
NVDA	0.044	0.052	0.066	0.284
ORCL	0.071	0.025	0.053	0.043
PM	0.115	0.080	0.043	0.099
SBUX	0.000	0.025	0.043	0.000
WBA	0.101	0.067	0.079	0.002
XEL	0.032	0.037	0.003	0.072

Table 4.2: Weights of portfolio assets for specific days.

4.2. The presented days were chosen because around March 16, 2020, a big drop in financial markets occurred. We can again notice the drop of BA stock weight from 0.056 on February 3, 2020, to 0.002 on March 9, 2020. The drop does not reach the depth of the one showed in Section 4.2, which can be caused by initial forecasts of volatility. They were slightly lower using EGARCH(1, 1, 1) than they were using GARCH(1, 1).

To sum up, this approach brought us gross depreciation of 2.17%. We got this number, the same way as we did in the previous section, by transforming log returns to standard returns by exponentiation them. Afterwards, the returns were multiplied with each other and finally subtracting the 1 and multiplying by

100 to get the percentage. Surprisingly it is a slightly bigger loss than in the previous case. We expected to get better results than in the previous case using GARCH(1, 1). However the difference is only 0.02 %, so we can not say which method is better to use during a financial recession.

4.4 Optimizing of the portfolio using mean 95 % ES

This section follows the steps taken in the previous Sections 4.2 and 4.3. This time the forecasted volatility on the diagonal of the covariance matrix is replaced by 95 % ES for each stock instead of using GARCH and EGARCH methodologies. This way we present a more cautious approach to the risk and we seek safer investment options. Again, in this case, we use MATLAB's built-in function *quadprog* to solve standard quadratic programming problems, and to get expected log returns we use EWMA method with $\lambda = 0.94$. This way we get the effective frontier, see Figure 4.13. We can see its concave plot ending at the point with the maximum possible return, receivable by investing all the resources in the stock with the highest expected return. The highest Sharpe Ratio is chosen, again see Figure 4.13. Compared with 4.3, it can be noticed that the maximal risk now is much higher. This is caused by using 95 % ES instead of variance. Nevertheless, the shape of the effective frontier is almost identical. The change in the weights can be seen in Figure 4.14. It is almost identical to the change in weights in Figure 4.4. This is caused by the fact that the variance and ES of each stock is highly correlated. The change of the risk of our portfolio can be seen in Figure 4.15. This time we do not observe two peaks as it was in Sections 4.2 and 4.3 but systematical incline of the risk over time instead.

In Figure 4.16, we can see how the estimated Sharpe Ratio changed over time. The peak around March 20, 2020, can be noticed, followed by a massive drop. This drop was explained in previous sections by peaking volatility at that time. However, in Figure 4.15 no such peak can be observed. Therefore, during that time expected log returns had to drop as well. Later on, Sharpe Ratio keeps on a slow decline, where on the other hand risk keeps fast incline. Therefore, the expected log return had to be on the rise during these days.

Finally, the change of weights of the portfolio assets over time can be seen in

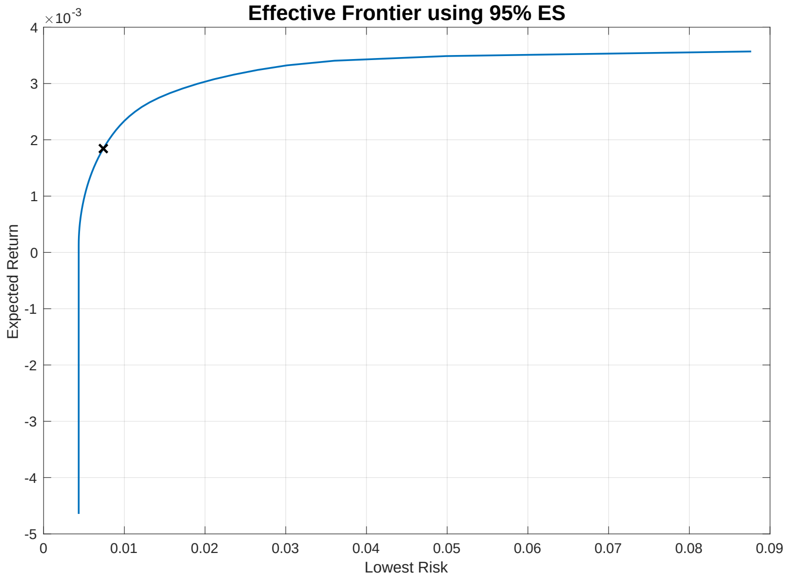

Figure 4.13: Effective Frontier - February 3, 2020
Where black cross represents the portfolio with the maximum Sharpe Ratio.

Figure 4.17. It can be noticed that Figure 4.17 is almost the same as were those in Sections 4.2 and 4.3. Therefore, we expect the total profit to be very similar to that in Sections 4.2 and 4.3. The table with estimated weights of assets in portfolio can be seen in Table 4.3. We can again notice the drop of BA stock weight from 0.055 on February 3, 2020, to 0.001 on March 9, 2020. Interestingly, the portfolio on June 24, 2020, is slightly more diverse than the one in Table 4.2 the same day.

To sum up, this approach brought us gross depreciation of 1.84%. These results are better than they were in Sections 4.2 and 4.3, which brings us to the conclusion that during an economic recession it is better to use more robust risk metrics such as ES. In the following section, we present a different approach but still working with ES.

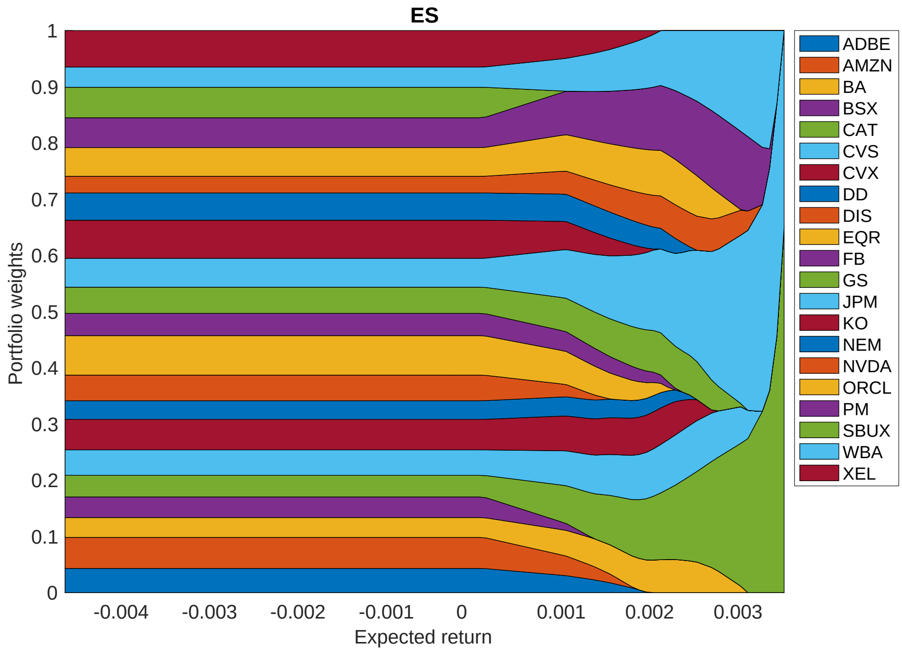

Figure 4.14: Weights of portfolio assets on the Effective Frontier

Figure 4.15: Risk of the portfolio over the time

Figure 4.16: Sharpe Ratio over the time.

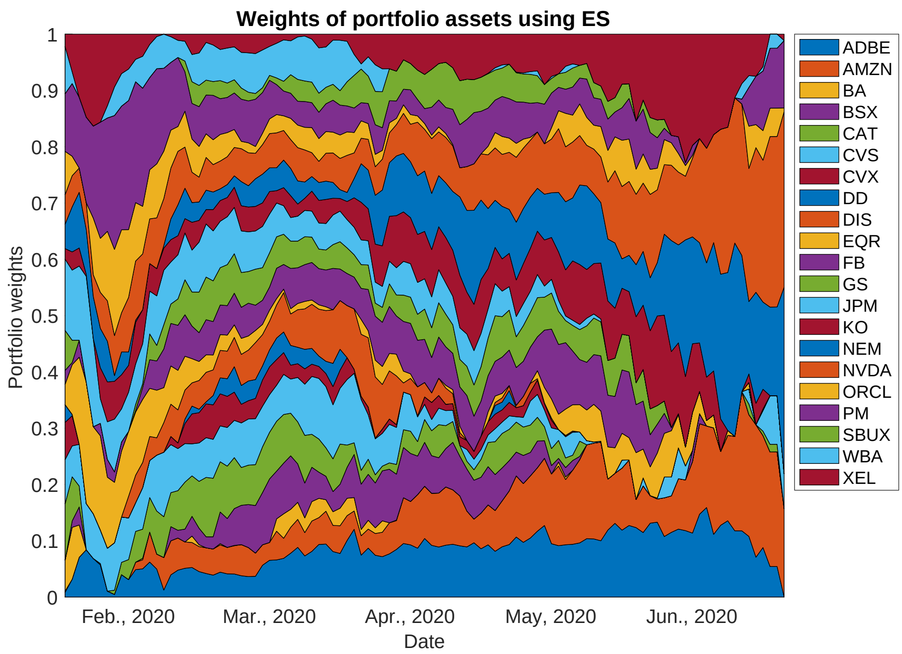

Figure 4.17: Weights of portfolio assets over the time.

STOCK	Feb. 3, 2020	Mar. 9, 2020	Mar. 16, 2020	Jun. 24, 2020
ADBE	0,008	0,041	0,073	0,088
AMZN	0,002	0,050	0,041	0,199
BA	0,055	0,001	0,026	0,000
BSX	0,000	0,065	0,062	0,000
CAT	0,100	0,091	0,085	0,008
CVS	0,078	0,066	0,091	0,037
CVX	0,066	0,050	0,027	0,000
DD	0,031	0,045	0,035	0,000
DIS	0,000	0,052	0,072	0,000
EQR	0,036	0,022	0,013	0,003
FB	0,025	0,056	0,065	0,000
GS	0,071	0,070	0,050	0,000
JPM	0,127	0,082	0,047	0,000
KO	0,019	0,036	0,022	0,035
NEM	0,043	0,020	0,029	0,155
NVDA	0,052	0,050	0,057	0,252
ORCL	0,077	0,032	0,040	0,053
PM	0,103	0,065	0,046	0,106
SBUX	0,000	0,024	0,037	0,000
WBA	0,084	0,058	0,080	0,008
XEL	0,022	0,022	0,003	0,058

Table 4.3: Weights of portfolio assets for specific days.

4.5 Optimizing the portfolio using Mean-ES

In this section, we optimize the portfolio using Mean-ES model. This approach differs from the previous ones not only in the methodological part but in used MATLAB functions as well. We use MATLAB built-in object ***PortfolioCVaR*** and its functions. At first, we import the learning data consisting of time series of considered stocks log returns. And apply Mean-ES model to them using 95% Expected Shortfall. We repeat this step 100 times to get an effective frontier, see Figure 4.18. We can see its concave plot ending at the point with the maximum possible return, receivable by investing all the resources in the stock with the highest expected return. Afterwards, we find the highest Sharpe Ratio, again see Figure 4.18. The change in the weights can be seen in Figure 4.19, supporting the previous statement. It is clear that at first, the portfolio is more diverse minimizing the risk, but it becomes less diverse therefore riskier with a higher expected return. Figure 4.19 differs from the figures presented in Section 4.2, 4.3, or 4.4. This is caused by a different built-in approach in getting expected log return of the stocks, using the mean of all the previous log returns. As it was shown in Table 3.2, the most profitable portfolio on February 3, 2020, using this approach, is the one consisting only of NVDA stocks.

Next, we add a new observation from the testing data set to the training data set and repeating the previous steps. This way we get portfolio optimization for each day from February 1, 2020, to June 30, 2020, when NYSE was opened. Several plots are provided.

In Figure 4.20 we can see the change in expected shortfall during the time. Especially, around the 30th day, an enormous jump, caused by the global pandemic spread. During the following days, 95% ES stayed more-less stable. This is the difference compared with Figure 4.15, where the risk measure as well ES equally inclined over time. It may be caused by considering expected log returns to be the mean of all the previous log returns. This way we believe that expected log returns, which were more-less stable over time were deviated by the economic recession.

Next, we can see Figure 4.21, which shows the change of estimated Sharpe Ratio over time. Around the 30th day, we can observe a huge drop caused by a

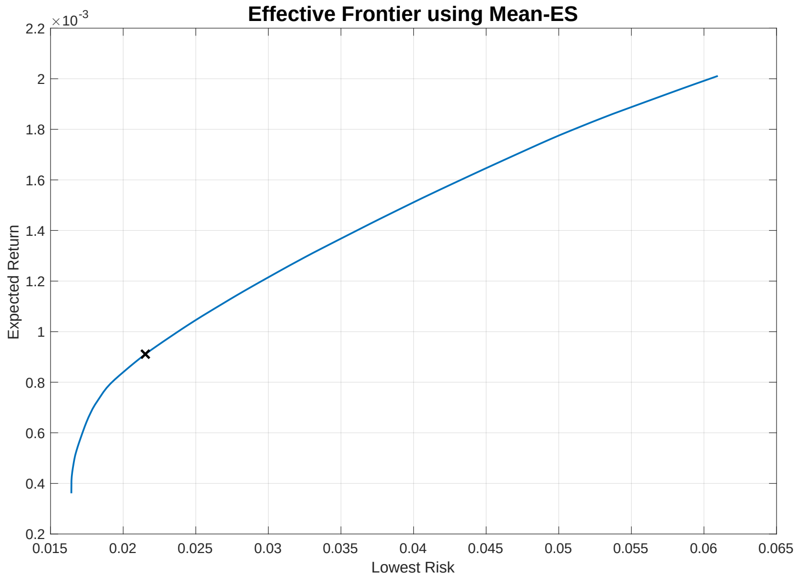

Figure 4.18: Effective Frontier - February 3, 2020
Where black cross represents the portfolio with the maximum Sharpe Ratio.

jump in the estimated 95% Expected Shortfall. Later we can notice that returns incline therefore Sharpe Ratio inclines as well.

Finally, we can look at Figure 4.22, where we can see a change in portfolio assets over time. Contradictory, to the previous approaches to portfolio optimization this one does not reflect the first all market shift in the middle of February 2020 as a reaction to WHO declaration of the COVID-19 outbreak as vividly. However, the second one in the middle of March 2020, which reflects a reaction to the countries closing their economies, is even more evident. We can see that for the first circa 30 days our optimal portfolio consisted mostly of 6 different stocks namely ADBE, AMZN, JPM, NEM, NVDA, and XEL. In the end our optimal portfolio consisted only of three different stocks AMZN, NEM, and NVDA. It is obvious that these companies were not affected by the global pandemic that much. NEM returns were almost independent in terms of correlation with other companies' returns. Since AMZN is an e-shop, its stock value raised, since it has been used even more during the global pandemic since people have to stay at home. And NVDA as a computer hardware company has not been touched

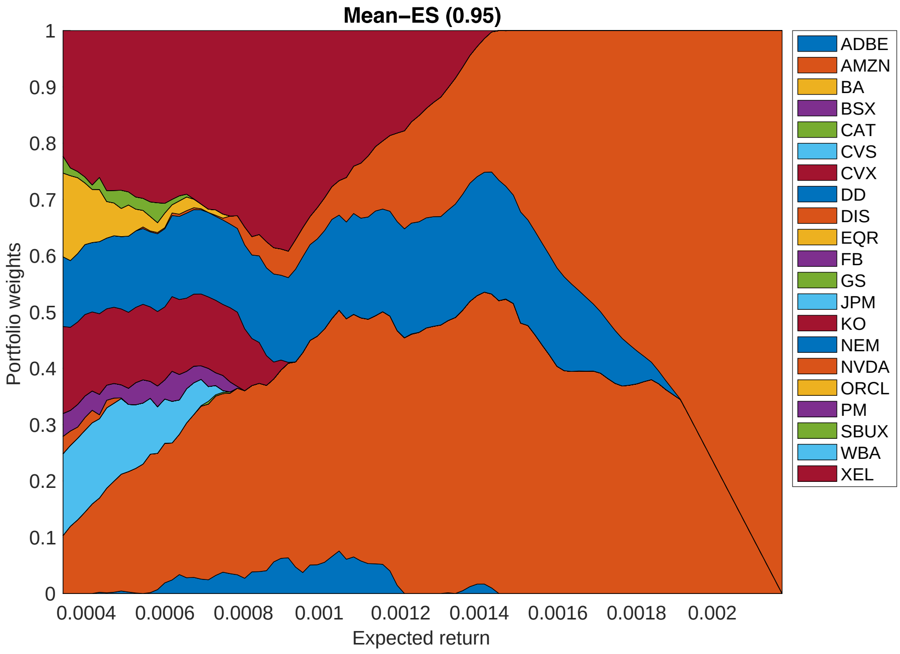

Figure 4.19: Weights of portfolio assets on the Effective Frontier

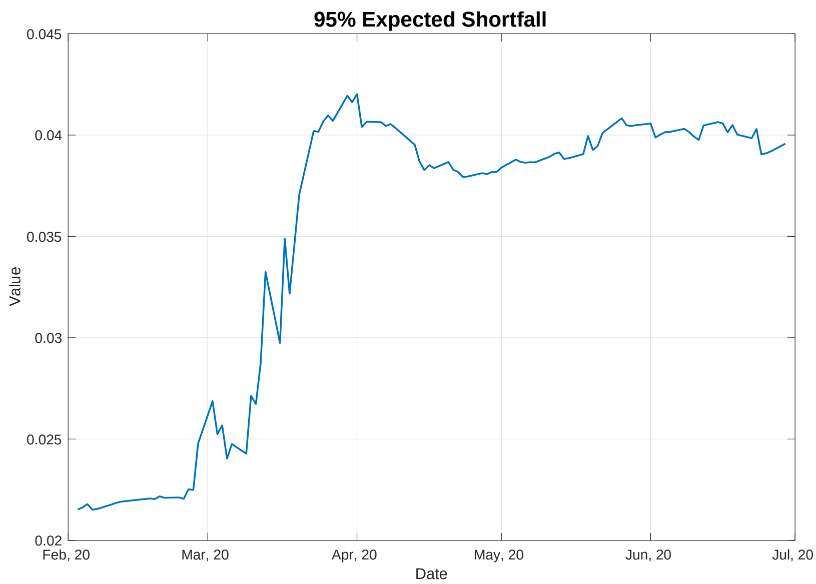

Figure 4.20: 95% Expected Shortfall over the time

Figure 4.21: Sharpe Ratio over the time.

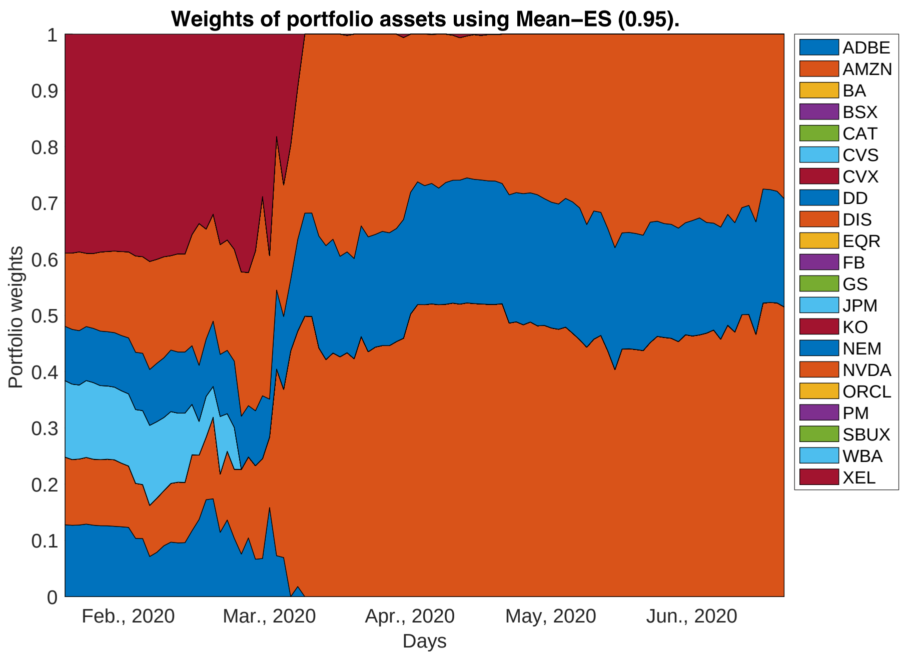

Figure 4.22: Weights of portfolio assets over the time.

STOCK	Feb. 3, 2020	Mar. 9, 2020	Mar. 16, 2020	Jun. 24, 2020
ADBE	0.127	0.104	0.000	0.000
AMZN	0.117	0.121	0.498	0.514
BA	0.000	0.000	0.000	0.000
BSX	0.000	0.000	0.000	0.000
CAT	0.000	0.000	0.000	0.000
CVS	0.000	0.000	0.000	0.000
CVX	0.000	0.000	0.000	0.000
DD	0.000	0.000	0.000	0.000
DIS	0.000	0.000	0.000	0.000
EQR	0.000	0.000	0.000	0.000
FB	0.000	0.000	0.000	0.000
GS	0.000	0.000	0.000	0.000
JPM	0.134	0.076	0.000	0.000
KO	0.000	0.000	0.000	0.000
NEM	0.097	0.117	0.184	0.192
NVDA	0.136	0.198	0.318	0.293
ORCLd	0.000	0.000	0.000	0.000
PM	0.000	0.000	0.000	0.000
SBUX	0.000	0.000	0.000	0.000
WBA	0.000	0.000	0.000	0.000
XEL	0.389	0.384	0.000	0.000

Table 4.4: Weights of portfolio assets for specific days.

by the global pandemic that much neither. What's more NVDA stock price was increasing rapidly lately since the increase of net revenue of the company. The table with estimated weights of assets in the portfolio can be seen in Table 4.4, the presented days were chosen because around the 30th the big shift in financial markets occurred. This observation is interesting because compared to the results from the previous sections, this methodology does not estimate as diverse portfolios. This fact is interesting since every methodology in this thesis was allowed to account for null weights for each stock. This may be caused by *PortfolioCVaR* object, using the historical method for calculating the expected return of each stock instead of EWMA method as in the previous sections. Nevertheless, it is a great opportunity for further research.

To sum up, this approach brought us a gross appreciation of 24.13%. We got this number by transforming log returns to standard returns by exponentiation them. Afterwards, multiplying the returns with each other and given weights and finally subtracting the 1 and multiplying by 100 to get the percentage.

All in all, even such a financial recession as occurred in 2020, this portfolio stayed in the profit and overcame every portfolio strategy mentioned before.

5. Conclusion and Comparison

The aim of this thesis was to compare different approaches to dynamic portfolio optimization using parametric models to model risk during the economic recession. At first, used methodologies were described. Followed by the empirical part, where were these approaches applied to suitably selected real data. In this case, the behaviour of portfolio optimization during the first half year of the global COVID-19 pandemic was examined.

From the empirical part is obvious that during the economic recession the markets, which are extremely volatile, make an opportunity for an investor to be profitable as well. The problem lies in setting up a risk measure, and the way this measure itself is modeled.

In Sections 4.2 and 4.3 portfolio optimization approaches were compared using the Dynamic Markowitz model, where the volatility of the stock's log returns was considered as a risk measure. The difference between these approaches lied in the methodology of modeling the volatility. In Section 4.2 GARCH model was used, resulting in the gross depreciation of the portfolio 2.15 %. In Section 4.3 EGARCH model, which takes leverage into account, was used. It was expected to end up better than the approach in Section 4.2. However, it resulted in the gross depreciation of the portfolio 2.17 %, and this way the depreciation of the portfolio is higher by 0.02 % than it was in Section 4.2. The difference is only 0.02 %, therefore we can not deduce that one approach is better than the other.

In Section 4.4 the same steps as in Section 4.2 were followed. However, this time the utilized risk measure was changed. Instead of volatility, we used 95 % Expected Shortfall. This was showed to work slightly better and resulted in a gross depreciation of 1.84 %. However, the weights of the portfolio were almost identical to the ones in Sections 4.2 and 4.3. Therefore, since the setting of the experiment was the same as in Sections 4.2 and 4.3 and the return we got was not dramatically better, we can only assume that it is a better option to utilize ES as a risk measure than volatility during the economic recession.

In Section 4.5, Mean-ES model for portfolio optimization was utilized. Compared to Markowitz model it is more sophisticated and computationally intensive method. Therefore we expected the results to overcome all of the three previously used approaches. Gross appreciation of the portfolio, when this approach was used, was 24.13 %. The interesting part is that *portfolioCVaR* object in MATLAB operates with expected returns that are calculated historically. This makes them quite stable and even when economic recession occurs they change their values only minimally. This way this model can not react to the change in expected return fast enough. In Sections 4.2, 4.3 and 4.4 EWMA method was used to get expected returns, making it possible to pursue opportunities to make a profit out of great volatility on the market. Therefore, it is even more surprising that the difference in the return was as immense. The other interesting point is that during the whole time, this approach invested only into a few safe stocks and did not diverse the risk among the other stocks. On the other hand approaches from previous sections got to this state only at the very end of the examined time window.

All of the approaches used, correctly detected the expected all market shifts especially the most obvious ones in the middle of February 2020 as a reaction to WHO, which declared the COVID-19 outbreak and the second one in the middle of March 2020 as a reaction to the countries closing their economies. However, the best approach to portfolio optimization during the economic recession was showed to be Mean-ES. Moreover, ES seems to be a better risk measure than the others. This was partially expected and also confirmed. There are still questions to be answered especially, whether by using Mean-ES model, which operates with expected returns calculated, not using the historical method, but rather using EWMA method, we can get an even better appreciation of the portfolio. We leave this question for future research as it would need to recreate the complete MATLAB object *portfolioCVaR*. The other interesting question lies in re-performing this study using Monte Carlo method on different data-sets making a statistically significant sample to decide which approach to portfolio optimization is all-round better.

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Acronyms

ARCH Autoregressive conditional heteroskedasticity

ARMA Autoregressive moving average

AVaR Average Value at Risk

Beta-Skew-t-EGARCH Beta Skew t EGARCH

CAViaR Conditional Autoregressive Value at Risk by Regression Quantiles

CCC-Garch Constant conditional covariance GARCH

CVaR Conditional Value at Risk

df Distribution function

EGARCH Exponential GARCH

ES Expected Shortfall

ETL Expected Tail Loss

EWMA Exponentially Weighted Moving Average

GARCH Generalized autoregressive conditional heteroskedasticity

GED Generalised Error Distribution

iid independent and identically distributed

NYSE New York Stock Exchange

P&L Profit and Loss distribution

VaR Value at risk

Var Variance

WHO World Health Organization

Symbols

L Loss function

Δ Time horizon

Ω Finite set of all natural states

Φ standard normal distribution function

α (VaR - ES) confidence level

R Real numbers

$\mathcal{A}$ Acceptance set

$\mathcal{G}$ Set of all risks

μ mean of loss distribution

ϕ probability density of standard normal

ρ Risk measure

σ standard deviation of loss distribution

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