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Autor: Marek Kolman

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Portfolio Credit Risk Modeling

(Modelování portfoliového kreditního rizika)

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Author: Marek Kolman

Supervisor: RNDr. Jiří Witzany, Ph.D.

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~ His school of thought opened another dimension of knowledge to my mind and irreversibly influenced my beliefs. ~

Declaration

I hereby declare that I am the only author of this thesis and furthermore I also declare that all external sources of information are marked.

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Marek Kolman

Abstract (English)

Thesis *Portfolio Credit Risk Modeling* focuses on state-of-the-art credit models largely implemented by banks into their banking risk-assessment and complementary valuation system frameworks. Reader is provided in general with both theoretical and applied (practical) approaches that are giving a clear notion how selected portfolio models perform in real-world environment. Our study comprises CreditMetrics, CreditRisk$^+$ and KMV model. In the first part of the thesis, our intention is to clarify theoretically main features, modeling principles and moreover we also suggest hypotheses about strengths/drawbacks of every scrutinized model. Subsequently, in the applied part we test the models in a lab-environment but with real-world market data. Noticeable stress is also put on model calibration. This enables us to confirm/reject the assumptions we made in the theoretical part. In the very end there follows a straightforward general overview of all outputs and a conclusion.

Keywords: credit risk, risk modeling, CreditMetrics, CreditRisk$^+$, KMV, Monte Carlo simulation

J.E.L Classification: C1, G3, G11

Abstract (Czech)

Diplomová práce *Modelování portfoliového kreditního rizika* se zaměřuje na moderní kreditní modely, jež si našly svá uplatnění v bankovních institucích a staly se neoddělitelnou součástí rámců řízení bankovních rizik. Čtenář získá znalosti jak na teoretické úrovni, tak i v praktické rovině o tom, jak vybrané modely fungují ve skutečném tržním prostředí. Práce do podrobna rozebírá modely CreditMetrics, CreditRisk$^+$ a KMV. V první části jsou teoreticky vysvětlena základní specifika, principy modelování a také je položeno několik hypotéz o silných/slabých stránkách modelů. Následuje praktická část, kde jsou modely zatěžovány v hypotetickém bankovním prostředí, avšak se skutečnými daty, přičemž je kladen velký důraz na modelovou kalibraci. To nám následně též umožní potvrdit či vyvrátit hypotézy, které byly položeny. Na samotném konci práce je poskytnut komplexní přehled výsledků spolu se závěrečným komentářem.

Klíčová slova: credit risk, risk modeling, CreditMetrics, CreditRisk$^+$, KMV, Monte Carlo simulation

Klasifikace J.E.L: C1, G3, G11

Contents

Compendium of used abbreviations

bp – basis point

BS – Black-Scholes formula

CAPM – capital asset pricing model

CDF – cumulative density function

DPT – default point

EAD – exposure at default

EDF – expected default frequency

DD – distance to default

EL – expected loss

FED – federal reserve

L – loss or liabilities

LGD – loss-given default

NYSE – New York Stock Exchange

PD – probability of default

PIT – point in time (type of approach)

PV – present value

RNG – random number generator

RR – recovery rate(s)

TTC – through the cycle (type of approach)

UL – unexpected loss

VaR – Value-at-Risk

WR – withdrawn rating

Introduction

> *When it is a question of money, everybody is of the same religion.*
>
> ~ Voltaire

Every day, human lives are tempted by influences driven by uncertainty. Such uncertainty generated by random events that we can ex-ante hardly predict, may cause radical *unexpected* changes in our lives as these variations are not incorporated into our future intents. It may, however, not be only negative but also positive unexpected occurrences. Generally, the futurity is interconnected with some risk, no matter whether positive or negative. It is apparently not only domain of human lives, since corporations face the same risk, just of a different pattern. Especially when we focus on long-term business relationships we notice substantial risks because one party is obliged to fulfil its obligation. The longer is the time horizon the higher extent of risk comes into play as the obliged party has to be not only *willing* but also *capable* to accomplish its agreed commitment. And exactly the importance of the capability factor increases drastically, the further we are looking ahead. Therefore in particular banks, whose entrepreneurship stands and falls on clients' redemptions of long-term promises, must keep an eye on risk management. There is plenty of auxiliary systems banks have developed to help them prudently control and monitor the risk. In this thesis we are going to scrutinize models that concern the core banking business, namely the credit portfolio risk models.

Reasonable credit management is a cornerstone for every classical bank. As a recent evidence, the great importance of credit policy has been proved when the economic crisis started to approach and is persisting up to the present time. If a bank's credit policy is too conservative, it is unnecessarily not fully using the market opportunities and hence the final profit is lower than it might be. On the other hand, being overly optimistic or aggressive in the credit management could be fatal for the bank, all the more so when there is a market downturn. To make it more complex, there are many other goals for a bank that the credit policy should comply with. It follows that – as has been just briefly described above – the credit portfolio management has to be governed close-to-perfectly to enable the bank do its overall banking business well. For the purpose of credit portfolio risk management, several models have been developed to assess the credit risk the portfolio is implying. It has been proved in last decades that banks relying on well-designed quantitative models show better

performance. In this thesis, we are going to focus on three of these models which finally became very popular and naturally widely implemented – CreditMetrics, CreditRisk$^+$ and at last the Moody's KMV[1] model. However, each of those models has some remarkable positives and naturally also some drawbacks that we also want to demonstrate.

To keep some logic in the thesis structure, it is divided into three principal parts. The first part is dedicated particularly to a description of the models. Reader will be informed on a theoretical level how the models operate and the main necessary assumptions are listed. The second part is rather practically oriented, which means the applicability of the models is demonstrated and verified on a real-world credit portfolio sample using a practical quantitative analysis. In that section it will also be explained how the models are calibrated, which is a crucial aspect of modeling since poorly calibrated models return misleading outputs. Consequently the results are compared. In the final summary, there follows an extract of the previous parts with a conclusion. All determinative and computed data are included in this thesis. Software applications that we use for the modeling needs are Wolfram Mathematica, VBA, SigmaPlot and SAS Enterprise. Occasionally we also use Matlab.

As it is clear from the description above, the explicit goal of this thesis is to describe, test and compare the selected models.

The reason of selection of this thesis-topic is simply a strong interest in (credit) risk management and what is more, we also want to discover models that are widely used in U.S. and other highly developed markets but which will most probably once penetrate also the Czech market - hence their way of functioning is not that known here in the Czech Republic but we may assume their great future.

[1] originally KMV was a stand-alone company but for generalization we use the contemporary trademark.

Part I

Theoretical part

1 Credit risk models

1.1 Lead-in

The necessity of credit risk management has been briefly justified in the introduction. Moreover, the credit risk might be more complicated and even more fatal than classical market risk and hence it also requires more vigilant attention. It stems from the nature of credit returns/losses. Market returns usually show the attribute of normality contrary to credit returns since credit returns are more asymmetric, which in practice means the returns are rather lognormally distributed as shown in Figure 1.

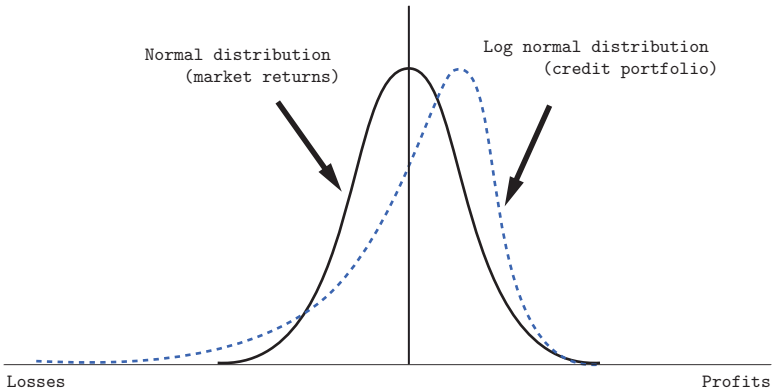

Figure 1: *Credit and market returns*

Portfolio credit risk models are models widely applied predominantly by banks to help them assess the credit risk they are exposed to since the point in time they grant a loan. When a loan portfolio is small it is observable in terms of single obligors and therefore it is beneficial to use a single-debtor deep analysis and screening afterwards. Contrarily, a large portfolio requires more sophisticated system to assess the risk because a large portfolio demands totally different approach as there are elements that are not to be found in a small portfolio. Just to imagine, assume a portfolio of only two debtors. It seems straightforward to rely on a deep individual-debtor analysis, since we can easily estimate the total risk if we know each obligor in detail. The second case with a large portfolio

is more complex. Without any doubts the single-debtor approach is neither effective nor reliable for a large portfolio. Moreover, not only it would be cumbersome and difficult to aggregate the results but what is more, it would also be costly in that case. Hence we need to use some portfolio-approach model to be able to measure the risk. Such models that rather than using a qualitative assessment make use of probability-based methods as their main engine. However, one has always to bear in mind that some sound lending policy is necessary. Even if the loans are being pooled in a large portfolio, the bank should perform a prudent pre-lending individual analysis. Better portfolio risk model does not mean a lower risk but better *estimation* of the risk. Since a loan has been granted, the real risk is the same in both[2] cases and the only difference resides in the accuracy of risk assessment.

It also needs to be noted that even a perfect model is not a clairvoyant. Risk management is therefore relatively easy to perform when the market is *stable*[3] but it is impossible to incorporate into a model unpredictable occurrence such as crisis. Banks should therefore rely on the model outputs with respect to "standard market conditions". In a longer horizon it might be helpful to monitor above all the macroeconomic data and re-calibrate the risk models as soon as such new data are at disposal.

1.2 Portfolio models

We are now going to focus more closer on the portfolio models as the main subject matter. There is a plenty of criteria which could be applied to the models segmentation. The models differ in a methodology that they use, sometimes significantly dissimilar assumptions are met. What is more, even their suitability is different as well as the outputs. Some of the models rely more on probabilistic theory rather than on given qualitative data etc. Following the instructions given in the introduction, we will scrutinize models CreditMetrics, CreditRisk$^+$ and KMV. Each of them is unique to some extent.

[2] bad/good portfolio model

[3] we intentionally write *stable* because even on a declining market but constantly declining with some stability it is easier to predict risks than on a stagnating market with severe double-sided fluctuations. Bad risk management in a narrow definition does not necessarily involve suffering large losses, it involves incorrect prediction of losses or even profits. (However, it is clear that bank usually does not want to suffer any loss, let alone large ones)

CreditMetrics represents a comprehensive tool suitable especially for large portfolios of business loans, it uses a simulation to generate scenarios eventually merged into a distribution. CreditMetrics is able to employ capital market data to provide more accurate results, it can also be supplemented with some additional modules to capture behavior of the real market. As a slight drawback relatively extensive inputs are required to enable the model provide credible results.

CreditRisk$^+$ is purely probabilistic model. It uses largely actuarial methods to estimate the losses of a credit portfolio. It simply takes into account probability of default (PD) of each obligor and based on exposure and a minor sector analysis it returns a distribution of potential losses. It usually performs well for retail and homogenous credit portfolios distinguished by a low probability of default and common characteristics of debtors. It requires no simulation.

Last but not least, **KMV** model has been chosen to be analyzed. This model is based on an in-depth analysis of capital structure of every obligor, trustworthy capital market data are therefore essential. The main benefit of in this way constructed KMV model is a great predictive power of obligor's default. The primary principle of this model is the option-pricing theory which in its modified form utilizes the capital market data to give us ex-ante signals about debtor's health. The KMV model in a plain version is intended to be used for individual-debtor monitoring. However, after an extension it can also provide results related to a credit portfolio. As the methodology resides in obligor's capital structure assessment, it is not applicable to retail debtors or debtors with no data recorded on capital market.

Having briefly outlined selected portfolio models, let us now focus in detail on deeper analysis. We will start chronologically with the CreditMetrics model.

2 CreditMetrics

2.1 Model overview

CreditMetrics has been developed by American bank J.P. Morgan in 1997 (authors: G. Gupton., C. Finger, M. Bhatia). It belongs to so called *mark-to-market* models, denoting it captures the market dynamics, represented by Markov process which is usually replicated by a transition matrix designed by a rating agency. Portfolio consisting of risky claims[4] is evaluated subject to debtors' qualities represented by their ratings at the end of a specified period. Under the rating migration process there is embedded Merton model, which is implicitly used by CreditMetrics but not scrutinized as in the case of KMV model, which goes more deeply into this subject. CreditMetrics simply assumes the rating reflects well the capital structure of each debtor and hence there is no need to simulate possible capital structure as KMV does. CreditMetrics also incorporates dependencies of debtors, which might be e.g. some common systematic factors. These dependencies are quoted as a pair wise correlation, which is subsequently translated into a correlation matrix indicating pair wise correlations throughout the whole portfolio. CreditMetrics is therefore a model rewarding well diversified portfolio with a lower riskiness[5]. The model can be upgraded with several additional modules that for instance incorporate volatility of recovery rates or negative relation of probability of default and recovery rates. After all, it follows that the output of the model is a distribution of simulated portfolio values. Simulation method used in this model is *Monte Carlo*.

As it is clear from the model description, this model complies with risk assessment of corporate claims portfolio. It is enough robust to grasp the main market dynamics and at the same time it does not require capital market information as KMV. On the other hand, KMV is more reactive since it tracks everyday capital market information whereas CreditMetrics relies solely on credibility of rating which is furthermore updated in some intervals and not on a daily basis.

[4] henceforward we will assume bonds

[5] the riskiness does not automatically mean that the bank suffers higher losses when the portfolio correlation is *ceteris paribus* higher. Riskiness represents rather the "connectedness" of the total result. With *ceteris paribus* high correlation on the one hand we can suffer higher losses than in uncorrelated portfolio, on the other hand there is also a chance of higher profit if the systematic factor pushes the quality of debtors. It therefore depends on bank's credit policy which one of those approaches it favors.

Having reviewed the crucial features of CreditMetrics let us take closer examination of the model and portfolio credit risk modeling in its environment.

2.2 Forward rates and forward values determination

The first target point in modeling by CreditMetrics we need to reach, is to determine possible future values of risky claims that the bank holds in its portfolio. To be able to do it, we firstly need to take several sub steps. Let us define the portfolio of J obligors such that $j = 1, 2, \ldots, J$ and further suppose that every obligor j has only one commitment to the bank (i.e. only one bond $B^{(j)}$). Every bond $B^{(j)}(t, T, s)$ can be evaluated at time t with respect to future rating s, for $T > t > t_0$ as follows

$$B^{(j)}(t, T, s) = \sum_{t_i=t}^{T} \frac{CF_{s,t_i}}{(1 + r_s)^{T-t_i}}, \tag{2.1}$$

where r_s is the forward discount rate (Lyuu, 2004). In case we do not know r_s, we can derive it by decomposing a long-term yield curve for s-rated zero-bond maturing in n years applying following formula

$$(r_n + 1)^n = (r_{0,m} + 1)^m (r_{m,n} + 1)^{n-m}, \text{where } r_{m,n} = r_s, \tag{2.2}$$

$n > m$ and where m, n denote beginning of the period and its end, respectively (Johnson, 2004). Those long-term zero-bond yields are publicly quoted, sometimes also denoted by spreads over a risk-free rate for the same time horizon.

Subsequently, for every single obligor we need to determine the value of bond for every rating grade. It involves calculating following vector of j-obligor's bond values, expressed as

$$\mathbf{B^{(j)}} = (B^{(j)}(t, T, Aaa), B^{(j)}(t, T, Aa), ..., B^{(j)}(t, T, default)), \tag{2.3}$$

where strings $Aaa, Aa, \ldots, default$ denote precisely rating s grade at time t.

When the possible forward bond values for the whole portfolio are calculated, we need a mechanism that enables us to attribute the possible bond values (or ratings from transition probabilities acquired from the transition matrix) according to some conditions. The process is described in the following chapter.

2.3 Implicit Merton model and rating migration thresholds

Before we turn our interest to rating migrations it is very helpful to clarify basic theory that stands behind the rating migrations.

Let us assume a notional company with assets value A, equity E and debt D. It holds that the value of A corresponds to the value of liabilities, denoted by $E + D$. We can also write $E = A - D$ to denote E as a cushion of capital over debt D. Now suppose the debt becomes due. As the debt is paid back from assets, the company must satisfy $A \geq D$ to be able to repay the outstanding debt. If $A \geq D$ is not satisfied by the due date, the company bankrupts[6]. If we assume the debt D to be constant within the debt due date, the only (non)default deciding factor is the asset value A which is not assumed to be constant but volatile. Having also decomposed $A = E + D$, where D is fixed constant, the volatility of A causes also the change in value of E. And this principle is the main idea of Merton structural model that defines E as a security cushion over D that protects the company from default as $A(t)$ is volatile and fluctuates (Cossin and Pirotte, 2000). Let us demonstrate the Merton model on a simplified scheme on Figure 2. It should be noted

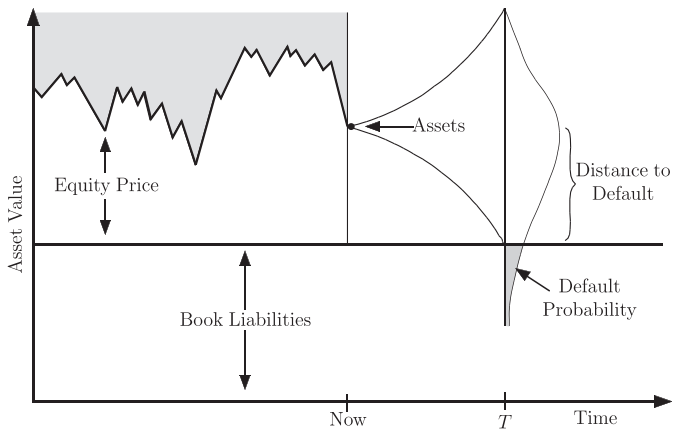

Figure 2: *Merton model. Source (Duffie and Singleton, 2003)*

that the asset value A is assumed to follow the geometric Brownian motion expressed precisely as (2.4)

$$dA = \mu_A A dt + \sigma_A A dW, \tag{2.4}$$

where μ_A denotes asset value drift, σ_A denotes asset volatility and $\sigma_A A dW$ is a randomly

[6] or more precisely the company defaults which is a subordinate term for bankruptcy. If a company bankrupts it definitely also defaults but when it defaults it does not automatically imply a bankruptcy

generated noise driven by Wiener process taking into account values of σ_A and A.

Merton model is rigorously analyzed particularly in KMV model as it is KMV's main component, whereas CreditMetrics adopts solely just the main principle and based on that it suggests following technique to set up the rating thresholds. For CreditMetrics, we should observe especially one important aspect. The higher is the value of A_1, the more "distant" the default is and the better the company performs when we abstract from other indicators. It can also be seen that there is some drift path, which is assumed to be the mean path when $\sigma_A = 0$. Hence it can be determined to be as the most "probable" that the company asset value A_1 at due time $T = 1$ will end somewhere close to the point determined by the zero asset volatility drift path. And exactly this principle is used to determine the migration thresholds.

Looking at a transition matrix (see for instance Table 2 in Section 7.1.2), we notice that one future probability for rating grade s at a given present time is usually much higher than the other ones. Such value is determined by the approximate estimation of A_1, the most probable scenario for A_1 is reflected by the highest probability of new rating. This principle helps us to derive the complete future rating thresholds for a given debtor. As we suppose the asset path is also driven by some arbitrary motion dW having the mean value 0, we suggest distribution that also fulfils this criterion. It is straightforward to use the standard normal distribution denoted by $N(0, 1)$ for setting up the thresholds at a given probability.

Knowing all the probabilities of migration, it is straightforward to compute the thresholds Zi by taking following steps (Gupton et al., 1997)

$$\begin{aligned}
&Z_D = \phi^{-1}(\Pr[s\,(T) = D \mid s\,(0)]),\\
&Z_C = \phi^{-1}(\Pr[s\,(T) = C \mid s\,(0)] + \Pr[s\,(T) = D \mid s\,(0)]),\\
&\vdots\\
&Z_{Aaa} = \phi^{-1}(\sum_{i=D}^{Aa} \phi(x_i)),
\end{aligned} \tag{2.5}$$

where $i = \{D, C, B, Ba, Baa, B, A, Aa, Aaa\}$ in the given order. Then we can set up rating

categories based on thresholds in a following way

$$\begin{aligned} &\forall z \in (-\infty; Z_D\rangle : s(T) \leftarrow {}'\mathrm{D}', \\ &\forall z \in (Z_D; Z_C\rangle : s(T) \leftarrow {}'\mathrm{C}', \\ &\vdots \\ &\forall z \in (Z_{Aaa}; +\infty) : s(T) \leftarrow {}'\mathrm{Aaa}', \end{aligned} \tag{2.6}$$

It is necessary to take this step for all $s^{(j)}(0) = \{C, B, Ba, Baa, B, A, Aa, Aaa\}$[7], so that we obtain full scope of thresholds for all initial ratings $s^{(j)}(0)$. Now, we are enabled using sampling from $\phi^{-1}(0,1)$ to determine future rating of every bond and from the Section 2.2 we also know how such a bond can be discounted. Last components we need to know to perform the simulation, are the portfolio correlations.

2.4 Incorporation of the correlations into CreditMetrics and Cholesky decomposition

The estimation of correlations is a known modeling bugbear because it is essential for a good calculation but it is certainly difficult to determine. We need to know the correlations to be able to incorporate dependencies of joint transitions. There are two basic approaches how to calculate or at least estimate pair wise correlations between the debtors. The first approach argues it is reasonable to separate the total sensitivity to market movements using a factor analysis that decomposes the total influence into a systematic component and idiosyncratic, i.e. firm-specific constituent (Witzany, 2010). Then it necessary to determine the weights of each of those factors and further for each obligor.

General formula for standardized return of j-debtor is given by

$$r_j = \sum_{i=1}^{k} w_i r(I_i) + w_{k+1}\varepsilon_j, \tag{2.7}$$

where $\sum_{i=1}^{k+1} w_i^2 = 1, r(I_i)$ is the standardized return on systematic factor I_i and ε_j is a firm-specific (idiosyncratic) factor (Witzany, 2010). If there are only two companies A, B and return of each of them is driven by only one systematic factor then the correlation $\rho_{A,B} = \rho(r_A, r_B)$ takes relatively simple form $\rho(r_A, r_B) = w_{1,A} \times w_{1,B}$[8]. The problem however

[7] the default state is omitted intentionally since we suppose there is no defaulted bond in the beginning. Besides, the default state is a finite state.

[8] even though there is an idiosyncratic factor for each company, their mutual correlation is 0, therefore it does not appear in the calculation.

becomes more complex as we suppose there are more systematic factors for each company since there is necessary to standardize the returns by calculating the index volatility. We provide just a succinct outline, to sketch the main idea but more deep concern is not an item of interest for this thesis.

When we have simple return form such as

$$R(I) = \sum_{i=1}^{k} w_i(I_i), \tag{2.8}$$

we need to standardize it into the form as in the formula (2.7). For such a calculation we need to determine volatility $\sigma(R(I))$ of the standardized return $R(I)$. In turn we have

$$\sigma(R(I)) = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \ldots + w_k^2\sigma_k^2 + 2w_1w_2\rho_{1,2}\sigma_{1,2} + 2w_2w_3\rho_{2,3}\sigma_{2,3} + \ldots + 2w_1w_k\rho_{1,k}\sigma_{1,k}}, \tag{2.9}$$

and we can now determine the rescaled, standardized return

$$r(I) = \frac{w_1\sigma(R(I_1))}{\sigma(R(I))} + \frac{w_2\sigma(R(I_2))}{\sigma(R(I))} + ... + \frac{w_k\sigma(R(I_k))}{\sigma(R(I))} = \sum_{i=1}^{k} \frac{w_i\sigma(R(I_i))}{\sigma(R(I))}, \tag{2.10}$$

and then it is finally possible to use the same correlation calculation approach as written above for the standardized return form (Gupton et al., 1997). However, this method is cumbersome and might also tempt to make mistakes therefore we suggest to use capital market data for correlations estimation if possible.

The second one, rather empirical way is to determine correlations from capital market data. It however requires credible time series data and diligent calculations but at least the results are based on real performance of the companies. Even the data series from stock market need not have to be perfectly reliable but it gives us at least approximate indication. One can also take into account *market Beta* (β) from *CAPM* to verify if the correlations are accurate.

Now assume having the following correlation matrix

$$\left.\begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1J} \\ \rho_{21} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{J1} & \cdots & \cdots & \rho_{JJ} \end{pmatrix}\right|_{\rho_{11}=1,\rho_{JJ}=1}. \tag{2.11}$$

When having the correlation structure determined, we can demonstrate how the correlations affect the calculation. Solving it analytically, we need to define joint migration probabilities for specified rating grades.

Let

$$f(r_A, r_B, \rho_{AB}) = \frac{1}{2\pi\sqrt{1-\rho_{AB}^2}} e^{-\frac{1}{2\pi\sqrt{1-\rho_{AB}^2}}({r_A}^2 - 2r_A r_B + {r_B}^2)} \tag{2.12}$$

be the probability density function of joint bivariate normal distribution for correlated returns r_A, r_B (it can also be translated so that it implies values triggering rating migrations) of debtors A and B, respectively. Then the joint cumulative density function for joint migration is expressed by formula (2.13) which is graphically represented by Figure 3.

$$\Pr\left[\mathrm{Z}(\mathrm{s}_{\mathrm{A},1}) < \mathrm{Z}_{\mathrm{A}} < \mathrm{Z}(\mathrm{s}_{\mathrm{A},2}) | \mathrm{Z}(\mathrm{s}_{\mathrm{B},1}) < \mathrm{Z}_{\mathrm{B}} < \mathrm{Z}(\mathrm{s}_{\mathrm{B},2})\right] = \int_{\mathrm{Z}(\mathrm{s}_{\mathrm{A},1})}^{\mathrm{Z}(\mathrm{s}_{\mathrm{A},2})} \int_{\mathrm{Z}(\mathrm{s}_{\mathrm{B},1})}^{\mathrm{Z}(\mathrm{s}_{\mathrm{B},2})} f(r_A, r_B, \rho_{AB}) dr_B dr_A. \tag{2.13}$$

Note that number 2 denotes rating at time state 2, number 1 marks rating at the initial

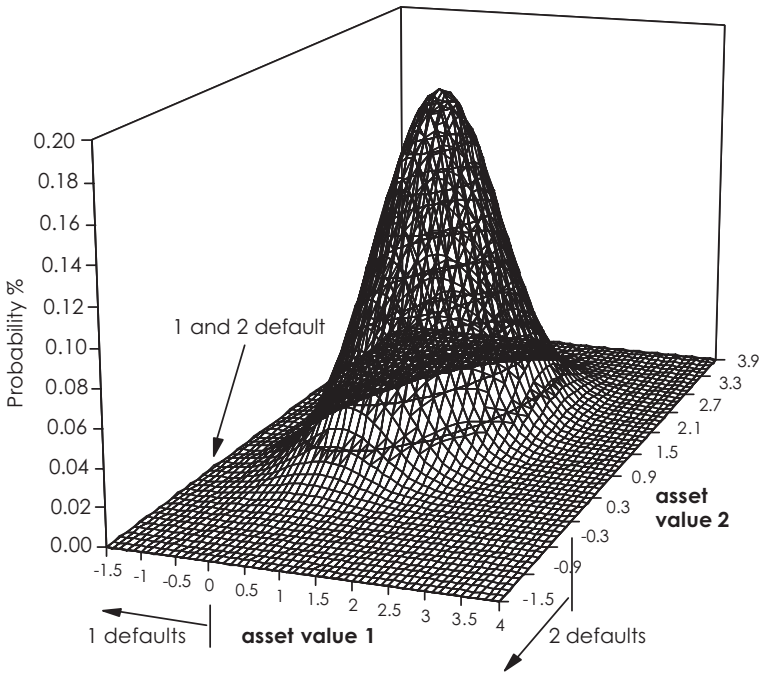

Figure 3: *Joint probability of migration (default). Source (Garside et al., 1999)*

state and for both A, B debtors in this case the joint probability of preserving their initial rating is expressed. This approach is however computationally inefficient when applied to

a larger portfolio.

Therefore it is better not to estimate the probabilities of joint migrations but to artificially simulate directly the correlated j-debtors' values $z^{(j)*} \sim N(0,1)$ triggering the transition and immediately obtain a scenario of newly assigned ratings $s_T(z^{(j)*})$, (subscript T stands for the desired finite time horizon) which we finally use to value the whole J-debtors size portfolio. A straightforward way is to use Cholesky decomposition of the correlation matrix to make us able to incorporate correlations to generated $z^{(j)} \sim N(0,1)$ finally denoted by $z^{(j)*}$, in words as a correlated rating migration values.

Let symbol $\mathbf{\Sigma}$ denote Cholesky matrix such that

$$\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^{\mathbf{T}} = \mathbf{L}\mathbf{U}, \tag{2.14}$$ [9]

whence

$$\mathbf{\Sigma} = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1J} \\ \rho_{21} & 1 & \cdots & \rho_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{J1} & \rho_{J2} & \cdots & \rho_{JJ} \end{pmatrix}\Bigg|_{\rho_{11}=1,\rho_{JJ}=1}, \mathbf{A} = \mathbf{L} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{J1} & a_{J2} & \cdots & a_{JJ} \end{pmatrix}$$

and (2.15)

$$\mathbf{A}^{\mathbf{T}} = \mathbf{U} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{J1} \\ 0 & a_{22} & \ddots & a_{J2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{JJ} \end{pmatrix}.$$

Matrix $\mathbf{A}$ is finally the desired necessary last input for Monte Carlo simulation and so we can perform it.

2.5 Monte Carlo simulation

Purpose of the simulation is to model a finite number of the portfolio values. The crucial principle of the simulation is to generate random scenarios which are subject to inputs we want to enter the model. To make the generated scenarios truly fair we have to use some

[9] $\mathbf{L}$ labels in a framework of a convetion the lower triangular matrix, whereas $\mathbf{U} = \mathbf{L}^{\mathbf{T}}$ denotes the upper triangular matrix. This notation is especially convenient for preventing the reader from ambiguity sometimes caused by notation $\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}$ where the matrix structure is not obvious.

trustworthy *random number generator* (RNG)[10] since non-randomly-generated scenarios would be skewed and the results hence not objective. Great emphasis is also attributed to the number of generated scenarios since higher amount of runs substantially reduces random errors and provides results with a higher confidence. The precision of Monte Carlo simulation goes as goes the inverse value of a square root of the total number of runs M, precisely as

$$precision = \frac{1}{\sqrt{M}}. \tag{2.16}$$

Our intention is to simulate portfolio values $V_1, V_2, \ldots, V_m$, for $m = 1, 2, \ldots, M$. Each V_m is determined by bond values $B^{(j)}_{m,T}(s^{(j)})$ at time horizon T, for J-debtors portfolio. $s^{(j)}$ denotes j-obligor's rating grade given by the simulation ordinal number m at time T.

Let us now specify more closer the simulation procedure. Let $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^{\mathbf{T}}$ be the Cholesky matrix containing the pair wise correlations and let $\boldsymbol{\theta} = (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(J)})^T$ be a column vector of J randomly generated numbers $\theta^{(j)} \sim N(0, 1)$. Then a modified vector of correlated numbers $\theta^{(j)}$ written in a vector form $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)}, \ldots, \tilde{\theta}^{(J)})^T$ is calculated as

$$\tilde{\boldsymbol{\theta}} = \mathbf{A}\boldsymbol{\theta}. \tag{2.17}$$

Performing this multiplication, we have obtained J-numbers $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)}, \ldots, \tilde{\theta}^{(J)}$, for which $\tilde{\theta}^{(j)} = z^{(j)*}$ which constitute randomly generated but correlation-adjusted numbers that need to be evaluated to assign new rating grade $s^{(j)}(z^{(j)*})$ to every debtor j in a given scenario. Such evaluation is based on a very simple method, we just use values $\tilde{\theta}^{(j)} = z^{(j)*}$ to assign rating category $s^{(j)}(z^{(j)*}) \in \{D, C, B, Ba, Baa, B, A, Aa, Aaa\}$ by treating values as z in the equation (2.6). Having the new rating categories $s^{(j)}_T$ for every obligor j, we can discount every debtor's bond following precisely the formula (2.1). Finally we have a vector of bond values subject to their new ratings $s^{(j)}$ at time T for all obligors[11]

$$\mathbf{B^{(j)}} = (B^{(1)}\left(T, s^{(1)}\right), B^{(2)}\left(T, s^{(2)}\right), \ldots, B^{(J)}\left(T, s^{(J)}\right)). \tag{2.18}$$

The value V of such bond a portfolio is simply a sum of bond values, expressed as

$$V = \sum_{j=1}^{J} B^{(j)}\left(T, s^{(j)}\right). \tag{2.19}$$

[10] for instance VBA RNG has a period of 2^{24}, plain RAND() function of Excel has a much lower period hence it is not suitable for simulations

[11] note that this outline relates to stand-alone scenario, therefore the subscript m is omitted for simplification

As we now know how to perform a single simulation, we further construct a loop to simulate[12] M scenarios to obtain $V_1, V_2, \ldots, V_m$, for $m = 1, 2, \ldots, M$. Having simulated all M values of the portfolio, we are allowed to analyze the results and assess the performance of our original portfolio.

2.6 Portfolio VaR

Portfolio values distribution might show very optimistic outcomes but however, our primary attention is to assess the risk which might be considerable even in otherwise well performing portfolio. Reasonable measure of portfolio risk is *Value-at-Risk* (VaR). Principle of VaR is to determine a quantile at some probability (confidence level) level α, written as $q_\alpha^{(VaR)}$, for which it holds that

$$\Pr\left[L \geq q_\alpha^{(VaR)}\right] = 1 - \alpha, \tag{2.20}$$

where L denotes the loss variable (Witzany, 2010). Usually, confidence levels α are chosen 95%, 97.5%, 99%, 99.5% or even 99.9%[13]. The difference between the 99.5% and 99.9% might seem to be negligible but one have to bear in mind that especially the credit portfolio loss distribution is "heavy-tailed" implying severer losses in the "tailed" distribution area than in the case of normal distribution. As a result, the absolute difference between $q_{99.5}^{(VaR)}$ and $q_{99.9}^{(VaR)}$ is largely very substantial.

To be able to determine $q_\alpha^{(VaR)}$ we can attribute some real probability distribution to our calculated one. The simplest way is to use $N(\mu, \sigma^2)$. Therefore we calculate the parameters as

$$\widehat{\mu} = \frac{1}{M}\sum_{m=1}^{M} V_m \text{ and } \widehat{\sigma} = \sqrt{\frac{1}{M-1}\sum_{m=1}^{M}(V_m - \widehat{\mu})^2}. \tag{2.21}$$

Then it is straightforward to estimate the $q_\alpha^{(VaR)}$. For example $q_{99.5}^{(VaR)} \cong \widehat{\mu} - 2.576\widehat{\sigma}$ meaning the value of portfolio based on our attributed normal distribution will not be lower than the mean portfolio value deducted of 2.576 multiples of standard deviation. For instance, for the level $\alpha = 99.9\%$ the numerical coefficient is 3.09 implying the $q_{99.9}^{(VaR)}$ will be relatively more distant and therefore severer than we might have expected from a 40bp shift in a confidence level.

[12] for instance applications such as Matlab or VBA can be used

[13] regarding Basel II regulatory framework, VaR is estimated at 99.9% confidence level whereas insurance Solvency II framework works with 99.5%.

We do not suggest such an approach to determine the portfolio risk since the loss distribution is not normally distributed. It might just be helpful for a rough estimation. It is also possible to test several hypotheses regarding distribution validity. E.g. for lognormality the *Kolmogorov-Smirnov*, *Anderson-Darling* or $Cramér - von - Mises$ tests. We need to get high p-value(s) in order not to reject the H_0 hypothesis that the tested sample originated from the tested distribution (Jondeau et al., 2006).

Empirically, it is better to determine true $q_\alpha^{(VaR)}$ values by ordinary quantile calculation based on sorting the sampled outcome by size and then determine the exact value of quantile of our simulated distribution. For example as $q_{99.5}^{(VaR)}$ would be chosen value whose size is on $0.005 \times M$-th placings when the scenario values are sorted in ascending order. In this way calculated VaR is then perfectly reliable in the framework of our simulation.

3 Modeling via CreditRisk$^+$

3.1 CreditRisk$^+$ as a model

CreditRisk$^+$ has been developed by Credit Suisse Financial Products (CSFP) in 1997. This model is totally different from the others portfolio models that we concern about to the intent that it neither processes classical market data nor it uses transition matrices, alternatively any other typical inputs for KMV or CreditMetrics. This model is based on an *actuarial* approach rather than on analyzing the causes or triggering default, therefore it is a type of reduced-form model. CreditRisk$^+$ claims that defaults are almost purely random occurrences and this idea is the main basis of how the input data are then treated by the model. This all however gives us a clear indication that the model is suitable especially for retail portfolios, where it may be complicated or inadequately costly to collect exact data and these portfolios should be also large in scale to show the power of probability theory in large samples to good advantage. CreditRisk$^+$ is therefore using *Poisson distribution* for predicting the default hence it also is marked as a *Poisson model*. The model is therefore classified as *default-mode* type since it recognizes only two final states of debtors, namely default and non-default.

Unlike CreditMetrics or KMV, there is also no simulation engine in CreditRisk$^+$ which just underscores the difference. The absence of simulation is not necessarily a drawback since the model is thereafter very fast and provides immediate results. To the contrary, the main drawback or rather a trade-off of no simulation is a poor environment to embody correlations which the model treats only via a relatively primitive sector analysis. Again, we can observe the model assumes common correlation within a sector not among individual debtors as such so that it fits for large and homogenous portfolios with low default probability of obligors. The model's inputs are **exposures**, **probabilities of default** and **default rates volatilities**. Output of the model is a distribution of possible losses, namely **credit VaR**.

In our practical part we test how the model performs on an inhomogeneous corporate portfolio, therefore we may not expect any accurate (or relevant) results in our case.

In the following chapters we are going to deeply focus on the CreditRisk$^+$ model's structure. It consists of three crucial parts: **default events**, **default losses** and a **sector analysis**.

3.2 Modeling events of default

Let us assume portfolio of J debtors such that $j = 1, 2, 3, \ldots, J$, then PD_j denotes default probability of every single obligor j. Now, we focus on a whole portfolio of J debtors. The PD in the whole portfolio of J debtors may be described well by a *generating function*

$$Q(t) = \sum_{n=0}^{\infty} \Pr\left[n \text{ defaults}\right] t^n, \tag{3.1}$$

where t denotes a supplementary variable. To be able to compute $Q(t)$ we need to know generating function of every j debtor. Therefore we set up $Q_j(t)$ to be j-debtor's generating function

$$Q_j(t) = 1 - p_j + p_j t, \tag{3.2}$$

as there only two possible states for every debtor, default and non-default. It is now necessary to "merge" the individual obligors into one portfolio. It is done by so-called *convolution*, typical element of fusing the generating functions. In a general formula the convolution takes a form of

$$\begin{aligned} &X_{A,B,\ldots,Z}(t) = Y_A(t) \cdot Y_B(t) \cdot \ldots \cdot Y_Z(t), \text{which is} \\ &X_{A,B,\ldots,Z}(t) = \prod_{i=A}^{Z} Y_i(t), \text{simply written as} \\ &X(t), \end{aligned} \tag{3.3}$$

where $Y_i(t)$ is a generating function for subjects $i = A, B, \ldots, Z$ and $X(t)$ is the convoluted generating function. Knowing the basis of convolution, we can return back to our model. In our case, following the formula (3.2) the convolution for $Q_j(t)$ is expressed as

$$\begin{aligned} &Q(t) = \prod_{j=1}^{J} Q_j(t), \text{which is after taking a logarithm} \\ &\log(Q(t)) = \sum_{j=1}^{J} \log(1 + PD_j(t-1)) \text{ and when } PD_j \text{ are small} \\ &\log(1 + PD_J(t-1)) = PD_j(t-1), \end{aligned} \tag{3.4}$$

since PD_j are small their powers can ne neglected. Moreover, the expression $\log(1 + PD_J(t-1))$ can be simplified because it in its limit form converges to

$$\begin{aligned} &Q(t) = e^{\sum_{j=1}^{J} PD_j(t-1)}, \text{which gives} \\ &Q(t) = e^{\mu(t-1)}, \text{ where } \mu = \sum_{j=1}^{J} PD_j. \end{aligned} \tag{3.5}$$

We have obtained nice-looking generation function $Q(t) = e^{\mu(t-1)}$ but we are also interested in the process that is underlying this function (CreditSuisse, 1997). Therefore we can perform *Taylor's expansion* of $Q(t) = e^{\mu(t-1)}$ to reveal the underlying distribution. Hence

$$\begin{aligned}
&Q(t) = e^{\mu(t-1)} \cong \xrightarrow{\text{Taylor's expansion of } e^{\mu(t-1)}} \cong \\
&\cong e^{-\mu} + e^{-\mu}\mu t + \tfrac{e^{-\mu}\mu^2}{2}t^2 + \tfrac{e^{-\mu}\mu^3}{6}t^3 + \tfrac{e^{-\mu}\mu^4}{24}t^4 + \tfrac{e^{-\mu}\mu^5}{120}t^5 + \tfrac{e^{-\mu}\mu^6}{720}t^6 + ... + \tfrac{e^{-\mu}\mu^n}{n!}t^n = \\
&= \sum_{n=0}^{\infty} \tfrac{e^{-\mu}\mu^n}{n!}t^n.
\end{aligned} \tag{3.6}$$

In the final summation, we can identify the Poisson distribution. The expression $\frac{e^{-\mu}\mu^n}{n!}$ is the probability of n defaults in the whole portfolio

$$\Pr\left[\text{n defaults}\right] = \frac{e^{-\mu}\mu^n}{n!}. \tag{3.7}$$

Note that the Poisson distribution is only an **approximation** of the real distribution, since for instance when we test on a notional small portfolio of 2 obligors the probability of 3 defaults using this expression, we still receive negligible, almost infinitesimal but still a positive probability[14]. It is also important to add that such occurrence in our model practically means the model admits more than 1 default for an individual obligor. For this reason we suggest revalidating the output data to filter out such results even though they are extremely rare.

In this chapter we have discovered the essential concept of how the defaults are modeled in CreditRisk$^+$. However, what interests us the most from the risk point of view is the possible suffered loss not the number of defaults even though those both occurrences are closely interconnected. For that reason in the following section we concentrate on the loss problem.

[14] to give a real support to this idea, suppose that $PD_1 = 0.01$ and $PD_2 = 0.02$, then the in fact unreal probability of 3 defaults of these 2 obligors according to the formula (3.7) equals $4.367 \times 10^{-6} = 0.000004367$ or 0.0004367%.

3.3 Implementing the loss attribute into CreditRisk$^+$

Knowing how the default events are driven, we now need to connect the j-debtor's default probability with a loss it is implying. Thus, we need to find a relation for loss L so that it holds (formally)

$$L = f(\text{n defaults}), \tag{3.8}$$

to enable us to set up probability-based loss function

$$L(n)^{Exp} = f(\Pr[\text{n defaults}]), \tag{3.9}$$

where $L(n)^{Exp}$ is the expected loss with respect to expected number of defaults n.

Since the Poisson distribution accepts only integers[15] as number of occurrences (defaults) n, we must take certain steps to allow enter only integers into the computation. This involves dividing the portfolio of debtors into M bands $m = 1, 2, \ldots, M$ by their exposure expressed in integers and subsequently we perform the same for expected loss (CreditSuisse, 1997). The bands will be denoted as $[m]$ and if j-obligor is placed in a band m, we write $j \in [m]$.

Let us now define

$$\begin{aligned} &\psi_j = B \times \nu_j \\ &\text{and} \\ &\zeta_j = B \times \varepsilon_j, \end{aligned} \tag{3.10}$$

where ψ_j denotes j-debtor's exposure expressed as integer multiple ν_j of unit amount B. In the same way ζ_j denotes j-debtor's expected loss expressed by integer multiple ε_j of unit amount B. Note that it is necessary to choose the value of B appropriately to make it possible to decompose the ψ_j and ε_j into integral multiples of B. Overly high value of B would also make the calculation inaccurate, nevertheless too low B value makes the calculation on the other hand excessively difficult[16].

Having collected data ν_j and ε_j for all debtors, we are now capable to assemble the exposure bands. Let $\varepsilon_{[m]} = \nu_{[m]} \times \mu_{[m]}$ be the expected loss in band m, where $\nu_{[m]}$ denotes the total exposure in m-th band and $\mu_{[m]}$ denotes number of expected defaults in band

[15] however it also accepts different inputs but in that case returns 0, which is of course not a desired result.

[16] or it might not be appropriate when we want particularly make use of the band approach.

m. The expected number of defaults in band m can be then expressed as

$$\mu_{[m]} = \frac{\varepsilon_{[m]}}{\nu_{[m]}} = \sum_{j \in [m]} \frac{\varepsilon_j}{\nu_j}. \tag{3.11}$$

We have derived the necessary items to be subsequently able to formulate a relation for the loss-probability function related to whole portfolio.

Define

$$U(t) = \prod_{m=1}^{M} U_{[m]}(t), \tag{3.12}$$

to be the loss-generating function for a portfolio consisting of M bands. Now the point is to look at each band as if it were a portfolio (Melchiori, 2004). Such an algorithm we have fortunately already deduced, see (3.6). In the case of bands the formula takes a form

$$\begin{aligned} U_{[m]}(t) &= \sum_{n=0}^{\infty} \Pr\left[\text{n defaults}\right] t^{n\nu_{[m]}} = \sum_{n=0}^{\infty} \frac{e^{-\mu_{[m]}} \mu_{[m]}^{n}}{n!} t^{n\nu_{[m]}} = \\ &= e^{-\mu_{[m]} + \mu_{[m]} t^{\nu_{[m]}}}. \end{aligned} \tag{3.13}$$

Again, for total loss in a portfolio, applying the convolution theorem (3.12) to (3.13) we obtain

$$\begin{aligned} U(t) &= \prod_{m=1}^{M} U_{[m]}(t) = \prod_{m=1}^{M} e^{-\mu_{[m]} + \mu_{[m]} t^{\nu_{[m]}}} = \\ &= e^{-\mu_{[1]} + \mu_{[1]} t^{\nu_{[1]}}} \times e^{-\mu_{[2]} + \mu_{[2]} t^{\nu_{[2]}}} \times \ldots \times e^{-\mu_{[M]} + \mu_{[M]} t^{\nu_{[M]}}}, \end{aligned} \tag{3.14}$$

which can finally be also expressed in an aggregated finite form

$$U(t) = e^{-\sum_{m=1}^{M} \mu_{[m]} + \sum_{m=1}^{M} \mu_{[m]} t^{\nu_{[m]}}}. \tag{3.15}$$

3.4 Default rates volatilities and sector analysis

In the previous chapter we managed to derive a formula for aggregate losses within a portfolio and hence we obtained a plain model. This chapter concerns with incorporation of additional influences and extending the one-sector plain model. Rather instead of including full correlations CreditRisk$^+$ substitutes the correlations with a methodology called **sector analysis**. It supposes debtors have some common specifics distinctive for a whole sector of alike debtors. Eventually, the sectors are treated as independent. The rationale underlying the sector analysis is the fact that default volatilities are driven by some systematic-risk factors, typical for every sector. Hence it is first needed to divide the obligors by sectors while we still respect the exposure bands, later on we can model default events or losses for every sector and finally we can merge those sectors' results

back into one portfolio applying the convolution theorem. The sector number of defaults as a variable $x^{(s)}$ is assumed to follow parametric **Gamma distribution** that is fully defined by its parameters, which are determined by volatility $\sigma^{(s)}$ and a mean value $\mu^{(s)}$. It will be proved that the convoluted function for number of defaults within a sector which originated from Gamma distribution can be transformed into a negative binomial distribution helping us to simplify the computation.

A weak point of the sector analysis in CreditRisk$^+$ is no clear suggested expert method how to form the sectors except the advice of using appropriate $\sigma^{(s)}$ and $\mu^{(s)}$ as benchmarks (Melchiori, 2004). The model only argues the volatility of default rates is based upon the principle of deviation of PD_j from the mean default intensity in the sector. It means that the model does not observe any real-world default volatilities but the volatilities are *de facto* implied by the model establishment and should not be applied outside CreditRisk$^+$. Therefore the model is also very sensitive on every modelers' judgement and also to the grouping. Nonetheless, the loose approach of the model to "sectorization" demonstrates it is not primarily aimed to inhomogeneous debts portfolios.

Let us now begin with specifying S-number of sectors for $s = 1, 2, \ldots, S$ and assigning the s-sector's number of defaults variable $x^{(s)}$, default rates volatility $\sigma^{(s)}$ and mean expected number of defaults $\mu^{(s)}$. For above quoted variables it holds that

$$\mu^{(s)} = \sum_{m=1|m\in s}^{M\in s} \frac{\varepsilon^{(s)}_{[m]}}{\nu^{(s)}_{[m]}} \text{ and } \sigma^{(s)} = \sum_{j\in m\in s}^{J\in m\in s} \sigma_j = \sum_{j\in m\in s}^{J\in m\in s} \frac{\varepsilon_j \sigma^{(s)}}{\nu_j \mu^{(s)}}. \tag{3.16}$$

The expected number of defaults in s sector $\mu^{(s)}$ is thus a sum of expected number of losses of the bands from which the sector is composed. Sector volatility $\sigma^{(s)}$ is expressed as summed volatility of every j debtor within the sector, signalizing there is no band-volatility.

Number of defaults variable $x^{(s)}$ is assumed to be driven by Gamma distribution. More precisely random variable $X \sim \Gamma(\alpha, \beta)$ has a density function of

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\beta}} x^{\alpha-1}, \text{ where } \Gamma(\alpha) = \int_{x=0}^{\infty} e^{-x} x^{\alpha-1} dx \tag{3.17}$$

is the Gamma function. α, β are parameters of $\Gamma(\alpha, \beta)$ and these parameters can be derived from the following expressions

$$\mu = \alpha\beta, \sigma^2 = \alpha\beta^2. \tag{3.18}$$

Hence after powering the first expression in (3.18) to

$$\begin{aligned} &\mu^2 = \alpha^2\beta^2, \text{ we derive} \\ &\alpha = \tfrac{\mu^2}{\sigma^2} \text{ and } \beta = \tfrac{\sigma^2}{\mu}. \end{aligned} \tag{3.19}$$

Knowing the underlying density function for $x^{(s)}$, we are now eligible to construct a generating function $Q(t)^{(s)}$ for number of defaults in sector s

$$\begin{aligned} &Q^{(s)}(t) = \int_{x=0}^{\infty} e^{x(t-1)} f(x)dx = \int_{x=0}^{\infty} e^{x(t-1)} \tfrac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\beta}} x^{\alpha-1} dx, \\ &\text{after a subtitution} \\ &Q^{(s)}(t) = \tfrac{\Gamma(\alpha)}{\beta^\alpha \Gamma(\alpha)(1+\beta^{-1}-t)^\alpha} = \tfrac{1}{\beta^\alpha(1+\beta^{-1}-z)^\alpha}, \end{aligned} \tag{3.20}$$

which can also be rewritten into a computationally convenient form

$$Q^{(s)}(t) = \left(\frac{1-p^{(s)}}{1-p^{(s)}t}\right)^{\alpha^{(s)}}, \text{ for } p^{(s)} = \frac{\beta^{(s)}}{1+\beta^{(s)}}. \tag{3.21}$$

Such a form however does not explicitly depict which process influences the probability of n defaults in the s sector (we only know how the generating function is designed). Therefore we can exercise the same approach that we have already once applied. More specifically, analogically (3.6) we decompose $Q(t)^{(s)}$ into polynomials using the Taylor's expansion as follows

$$\begin{aligned} &Q^{(s)}(t) = \left(\tfrac{1-p^{(s)}}{1-p^{(s)}t}\right)^{\alpha^{(s)}} \cong \xrightarrow{\text{Taylor's expansion of } \left(\frac{1-p^{(s)}}{1-p^{(s)}t}\right)^{\alpha^{(s)}}} \\ &\cong (1-p^{(s)})^{\alpha^{(s)}} + (1-p^{(s)})^{\alpha^{(s)}} p^{(s)}\alpha^{(s)}t + \tfrac{1}{2}(1-p^{(s)})^{\alpha^{(s)}}\left(p^{(s)}\right)^2(\alpha^{(s)} + \left(\alpha^{(s)}\right)^2)t^2 + \\ &+\tfrac{1}{6}(1-p^{(s)})^{\alpha^{(s)}}\left(p^{(s)}\right)^3(2\alpha^{(s)} + 3\left(\alpha^{(s)}\right)^2 + \left(\alpha^{(s)}\right)^3)t^3 + ... = \\ &= (1-p^{(s)})^{\alpha^{(s)}} \sum_{n=1}^{\infty} \begin{pmatrix} n+\alpha^{(s)}-1 \\ n \end{pmatrix} \left(p^{(s)}\right)^n t^n, \end{aligned} \tag{3.22}$$

in which it can be identified that $Pr[n\ defaults]$ is driven by **Negative binomial distribution**. It is therefore straightforward that the probability of n defaults in s sector is

$$\text{Pr[n defaults]} = (1-p^{(s)})^{\alpha^{(s)}} \begin{pmatrix} n+\alpha^{(s)}-1 \\ n \end{pmatrix} p^{(s)}. \tag{3.23}$$

The very last step to make the CreditRisk$^+$ model complete is to translate the distribution (or number) of defaults into losses and subsequently define a distribution of losses. The

ultimate distribution of losses going over all sectors s is described by a generating function

$$U(t) = \prod_{s=1}^{S} U^{(s)}(t), \tag{3.24}$$

where $U(t)$ is the final-loss probability generating function and also $U(t)^{(s)}$ denotes partial (sector)-loss probability generating function. To be able to determine unknown generating function $U(t)^{(s)}$ we can use a link function, namely $p(t)^{(s)}$ such that

$$U^{(s)}(t) = Q^{(s)}(p^{(s)}(t)), \tag{3.25}$$

in which

$$p^{(s)}(t) = \frac{\sum\limits_{m=1}^{M\in s} \left(\frac{\varepsilon_{[m]}^{(s)}}{\nu_{[m]}^{(s)}}\right) t^{\nu_{[m]}^{(s)}}}{\sum\limits_{m=1}^{M\in s} \left(\frac{\varepsilon_{[m]}^{(s)}}{\nu_{[m]}^{(s)}}\right)} = \frac{1}{\mu^{(s)}} \sum_{m=1}^{M\in s} \left(\frac{\varepsilon_{[m]}^{(s)}}{\nu_{[m]}^{(s)}}\right) t^{\nu_{[m]}^{(s)}}. \tag{3.26}$$

$p(t)^{(s)}$ is then evaluated over bands m. We can make the calculation make more complex when we incorporate an assumption of decomposing the bands into J single obligors. Then the $p(t)^{(s)}$ from (3.26) takes a full form of

$$p^{(s)}(t) = \frac{\sum\limits_{j=1|j\in m\in s}^{J\in m\in s} \left(\frac{\varepsilon_{j\,[m]}^{(s)}}{\nu_{j\,[m]}^{(s)}}\right) t^{\nu_{j\,[m]}^{(s)}}}{\sum\limits_{j=1|j\in m\in s}^{J\in m\in s} \left(\frac{\varepsilon_{j\,[m]}^{(s)}}{\nu_{j\,[m]}^{(s)}}\right)} = \frac{1}{\mu^{(s)}} \sum_{j=1|j\in m\in s}^{J\in m\in s} \left(\frac{\varepsilon_{j\,[m]}^{(s)}}{\nu_{j\,[m]}^{(s)}}\right) t^{\nu_{j\,[m]}^{(s)}}. \tag{3.27}$$

After several minor rearrangements we can transform $p(t)^{(s)}$ given in (3.27) into the desired form of $U(t)^{(s)}$ and its total form $U(t)$ derived in (3.25). That gives $U(t)$ in terms of $p(t)^{(s)}$ as

$$U(t) = \prod_{s=1}^{S} U^{(s)}(t) = \prod_{s=1}^{S} \left(\frac{1 - p^{(s)}}{1 - \frac{p^{(s)}}{\mu^{(s)}} \sum\limits_{m=1}^{M\in s} \left(\frac{\varepsilon_{[m]}^{(s)}}{\nu_{[m]}^{(s)}}\right) t^{\nu_{[m]}^{(s)}}} \right)^{\alpha^{(s)}}, \tag{3.28}$$

which is the final desired expression for loss distribution over all sectors S (CreditSuisse, 1997).

The model then proposes using *recurrence relations* to calculate the total loss. The rationale behind the recurrence relations is that they facilitate the calculations given by (3.28) using its logarithmic transformation and taking derivatives of such form. Then every expression is shown as a fraction of the first derivative and its non-differentiated form. After solving these fractions we obtain the final result.

The distribution of losses is of a VaR form, meaning to each single unit of loss is attributed a probability of occurrence implying the result is given in a different form than in the case of CreditMetrics and KMV where we obtain a distribution of the portfolio *value*. Note that as CreditRisk$^+$ takes into account only PD and exposures, the best-case scenario is just the eventuality when there is no default. The value of the portfolio cannot exceed its original value even if there is no default expected in contrast to CreditMetrics where the value was given by debtors' ratings whose employment is relaxed in case of CreditRisk$^+$. KMV is the same case as CreditMetrics in this issue which means the portfolio value can increase. Hence CreditRisk$^+$ does not assume the debt is going to be evaluated in order to be traded and thus it is not appropriate for corporate bonds which can be traded with ease. The bottom-line of the credit analysis in CreditRisk$^+$ is that it concentrates solely on the defaults and loss as such not on the other aspects of the portfolio quality.

4 The KMV model

Stand-alone KMV model is a plain cornerstone of a more complex risk analysis framework called KMV Portfolio manager which extends the environment of plain KMV to more obligors. We now focus on the plain KMV version, the portfolio supplement is described in the later chapters.

KMV (abbreviation of its founders' names S. Kealhofer, J. McQuown and O. A. Vasicek[17]) is a credit risk model, developed by KMV company in 1989. The model is primarily not determining value of debt but a probability of debtor's default[18]. It deeply monitors debtor's fitness by analyzing the company's **capital structure**. As mentioned in the Section 2 dedicated to CreditMetrics, KMV fully and explicitly uses **Merton model** to produce signals about debtor's probability of default. For this reason it also employs *Black-Scholes option pricing theory*, which is the main engine used to assess the debtor in KMV framework. Subsequently, taking into account the results from Black-Scholes formula, it attributes to every debtor **distance to default** (*DD*) statistic which is a dimensionless number[19]. Distance to default is therefore rather converted into main KMV output variable which is **expected default frequency** (*EDF*) to provide an indication which is easy to understand. KMV also constitutes it disagrees with probability of default based on a company rating. It essentially argues that the real probabilities of firms' transitions in rating are much more likely than those given by transition matrices produced by rating agencies. It is also important to highlight that it does not mean the ratings are inaccurate but they might be slow in reflecting the firm's true state[20]. Whereas KMV, processing firm's market data almost constantly, is much more responsive in determining firm's health. On the other hand, rating is not assessing only capital structure but also other aspects, while KMV concerns only the capital structure and does not take into account other influence(s). From the KMV's ability to determine own *PD* stems that there is also a possibility to create *KMV transition matrix*. In any case, empirically has KMV proved that its predicting power is on a very high level, e.g. default of *Enron* or *Lehman Brothers* was forecasted almost one year ahead in comparison to rating-based probabilities

[17] one of the best Czech mathematicians who emigrated to USA

[18] which can be, however, with ease applied to a risk-based debt valuation

[19] nevertheless the distance to default is certainly a good default indicator

[20] the rating-based debt pricing is rather of through the cycle (TTC) type whereas the KMV's market-sticky valuation is of point in time (PIT). See Appendix B for more details.

of default. Consequently, in default prognosis is KMV a market top product.

Let us return back to the model output. The *EDF* is a nice, largely subjective tool how to quantify the probability of firm's default but what we also need in scope of comprehensive risk management is to derive price of a risky claim whose issuer is the scrutinized obligor. And considering *EDF* as a real probability of default it is needed to apply **risk-neutral** (adjusted) discounting rates to determine the fair price of debt. In fact such an approach is named *martingale pricing*, which is supposed to be state-of-the-art way how to valuate risky instruments (Neftci, 2000). After having discounted the risky claim we can finally determine portfolio value, consisting of risky obligations.

In this KMV introduction we have outlined the main ideas behind the model, let us therefore move forward to the modeling within KMV framework itself.

4.1 Asset random walk, Merton model and payoff

The crucial point that KMV concerns are the **firm's assets**. The complete structure of the model is based on Merton model which deeply analyzes the asset drift and volatility. Consequently, similarly CreditMetrics, KMV states that company defaults when its assets A fall below debt level D at the due date T. Basically the very same scheme as in the Section 2.3 is applied. The only difference resides in the definition of debt D. Merton model generally defines debt as only one debt obligation (a zero-bond) with a face value of D and maturity at time T. KMV discovered by experience that such an arrangement is not optimal, since we can hardly assume the company debt is only financed by a single zero-bond. As a result, KMV proposes defining so called **default point** (DPT) instead of overall debt D, in the following way

$$\text{DPT} = \text{Short term debt} + \frac{1}{2}\text{Long term debt}, \tag{4.1}$$

which means the "importance" of long-term liabilities for default has been decreased by setting a lower weight in the total sum. Other aspect of DPT is the fact that it might change within the observed period but the calculation taking into account the ex-ante DPT assumes it to be constant. It is therefore also possible to add into the model[21] DPT oscillations so that it simulates some volatility. It strongly depends on how the firm

[21] we intentionally do not write "improve" since it does not need to be unconditionally true

operates, hence very individual. In any case, the debt level should be revaluated when there is a notion of change in debt.

Let us now focus more on the role of option theory in Merton model. If bank grants a loan it can be repaid *if and only if* the company has enough resources to do so otherwise it defaults on the payment. If we divide liabilities very roughly into debt D and equity E then it holds that $D + E = L$ (liabilities), and it also corresponds to $D + E = A$ (assets) since there is always a parity between assets and liabilities. The company is therefore solvent only if there are enough assets A to redeem the debt ($A \geq D$). If not (i.e. $A < D$), the company defaults. Now turn our interest to the payoff. There are equity holders claiming E and creditors (bank, granted loan of D). If $A \geq D$, the creditors receive entire D at the due date, whereas equity holders receive the surplus of A over D, which is $E = A - D$. The higher is the value of assets A, the higher is E (the debt D is constant) and hence the higher is the payoff for equity holders, so that the payoff for equity holders is virtually unlimited. If the company is not having enough assets to repay the debt ($A < D$) it firstly must allow to use entire assets A to at least partially pay the debt D and since there is no excess over A, the equity E is zero and equity holders receive zero payoff. Summarizing the payoff part we can formulate the payoff relations at the due date. Payoff for equity holders $E^* = Max(A - D, 0)$, which is the same as the *call-option* payoff and payoff for creditors (bank) $D^* = D - Max(D - A, 0)$, which is the same payoff as a in the case of *put-option*.

Since it is assumed the firm's assets follow *geometric Brownian motion* $dA(t) = \mu_A A(t)dt + \sigma_A A(t)dW$ as a stock in case of stock options, we can conveniently combine the option pricing theory with equity and debt valuation in a case of a company. Hence it is the moment when Black-Scholes formula (BS) comes into play.

4.2 Black-Scholes formula, Itô's process and company's assets

Our primary target is to derive default probabilities EDF[22] for a company utilizing the Black-Scholes formula. EDF can only be mapped when we know distance to default DD since these two indicators are uniquely interconnected. Distance to default stems directly from Black-Scholes formula but the problem is there are two variables whose values are not empirically observable on a capital market, that are $A(t)$ and σ_A. Therefore we need

[22] subsequently we also need to apply the risk-neutral adjusted EDF rates to bond pricing

to employ more rigorous option-related theory to imply these unobservable values. Let us now formulate some important relations to completely solve the primary problem.

Company's assets follow the geometric Brownian motion

$$dA(t) = \mu_A A(t)dt + \sigma_A A(t)dW, \tag{4.2}$$

where $A(t)$ is the asset value, μ_A is asset-value drift and σ_A is asset volatility. Note that because we use *martingale pricing* which is risk neutral, μ_A will be substituted by a risk-free rate r. As a next point, let us specify general Black-Scholes formula for a call option on non-dividend European stock (Hull, 2008)

$$\begin{aligned} &C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \\ &\text{where} \\ &d_1 = \frac{\ln\left(\frac{S}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}, \end{aligned} \tag{4.3}$$

where $C(S,t)$ is the call option premium, S is stock's spot price, K is a strike price, r is a risk-free rate, $T-t$ is a time to option's maturity, σ is stock's volatility and $N(.)$ denotes Normal distribution CDF[23]. We now adapt this classical formula (4.3) for our environment of a firm with respect to its capital structure. Hence

$$\begin{aligned} &E_0 = A_0N(d_1) - De^{-rT}N(d_2), \\ &\text{where} \\ &d_1 = \frac{\ln\left(\frac{A_0}{D}\right)+(r+\frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}} \text{ and } d_2 = d_1 - \sigma_A\sqrt{T}, \end{aligned} \tag{4.4}$$

where E_0 is the equity price, A_0 is the value of assets, D is DPT, σ_A denotes asset volatility and all is fixed at time $t = 0$ (Crosbie and Bohn, 2003). Note that this point of view is static contrary to (4.3). We can also notice the "strike price" K is substituted by D i.e. DPT which signals the threshold for bankruptcy and in this case also the turning point of payoff to equity holders.

The main problem regarding (4.4) is that we can not observe A_0 and σ_A on the capital market but at least the other variables are known. Having only one equation with two unknown variables we demand one more relation to solve a system of equations. **Itô's lemma** proves to be extremely useful for this case. If a variable x follows the *Itô's process*

$$dx = a(x,t)dt + b(x,t)dW, \tag{4.5}$$

[23] cumulative density function

which is in our case (4.2) then a process $E(A,t)$ (that is represented by (4.4)) according to Itô's lemma follows a process

$$dE = \left(\frac{\partial E}{\partial A} rA + \frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial E}{\partial A}\sigma_A^2 A^2\right) dt + \frac{\partial E}{\partial A}\sigma_A A dW. \tag{4.6}$$

Since there is a direct relation between E_0, σ_E and A_0, σ_A over *Greek delta*, we obtain an equation

$$\sigma_E E_0 = \frac{\partial E}{\partial A}\sigma_A A_0, \tag{4.7}$$

where $delta = \frac{\partial E}{\partial A} = N(d_1)$ and after a substitution for *delta* in (4.7) we get

$$\sigma_E E_0 = N(d_1)\sigma_A A_0, \tag{4.8}$$

which is finally the second equation to help us to determine A_0 and σ_A. System of equations (4.4) and (4.8) must be solved iteratively because there is no way how to explicitly express unknown variables A_0 and σ_A from d_1 over normal distribution. Note that A_0 and σ_A are in fact only DPT-proportionate values since we treat D as DPT. In (4.4) we can see the "strike price" is DPT hence only DPT-adjusted values are determinative for us. If we instead used the entire value of debt D the A_0 derived from (4.4) and (4.8) would be "too high" and σ_A "too low" so that the debtor would be very far from default and in modeling approach he would never default. Putting all together the A_0 and σ_A are not even trying to be close to real A_0 and σ_A values in our case, they are rather real values adjusted to DPT that is the default threshold.

4.3 Distance to default and real-world vs. risk-neutral probabilities

Knowing A_0 and σ_A, we can sample via *lognormal property* or model asset walk using discrete path modeling to obtain $A(T)$ (see (Witzany, 2010) for more details). The lognormal property states that $A(T)$ has a value determined by a formula (Hull, 2008)

$$\ln(A(T)) \sim N\left(\ln A_0 + \left(r - \frac{\sigma_A^2}{2}\right)T, \sigma_A^2 T\right). \tag{4.9}$$

Computationally it is however much more efficient to use the lognormal property since we do not need to concern time steps etc., which are needed in case of the discrete path modeling and it however also gives approximately the same result as well (Schmidt and Moser, 2004). As soon as we know $A(T)$, the distance to default DD can be calculated simply as (Sironi and Rosti, 2007)

$$DD = \frac{A(T) - DPT}{A_0\sigma_A}, \tag{4.10}$$

or we might use d_2 from the formula (4.4) since $DD = d_2$. DD implies expected default frequency EDF[24] so that we finally receive the desired probability of default as the model's output. The mapping mechanism can be formally written as

$$EDF_t = f(DD, t, \text{economic sector}), \tag{4.11}$$

but such an explicit relation is impossible to construct. In practice KMV company uses a large database [25] of defaulted firms with the same DD as the observed company has. Naturally, there is an inverse relation of DD and EDF[26]. But there are moreover some additional particularities. EDF is a real-world probability, which nicely shows how the firm in fact performs. Nevertheless, our intention is to price the risky bond with respect to risk, not only to determine the default probability. Hence, there is a complication since such a credit instrument must be adequately discounted using a risk-neutral probability in connection with martingale pricing. It involves translation of EDF into a risk-neutral probability PD_{rn} as (Crouhy et al., 2000)

$$PD_{rn} = N(N^{-1}(EDF) + \frac{\mu_A - r}{\sigma_A}\sqrt{T}). \tag{4.12}$$

It is obvious that if $\mu_A > r$[27], the price of risk is positive implying $PD_{rn} > EDF$, and conversely $\mu_A < r$ constitutes negative price of risk implying $PD_{rn} < EDF$. $\mu_A - r$ can be derived from equation (Crouhy et al., 2000)

$$\mu_A - r = \beta\pi, \tag{4.13}$$

where β is *beta coefficient* from the $CAPM$ model and π is a market risk premium $\pi = \mu_M - r$.

Knowing PD_{rn} we are eligible to discount a risky bond. Such a discounting process consists of **two parts** since we discount separately the risk-free and risky part of possible bond's cash flow. The idea behind this "separated discounting" is simply stemming from safe amount of recovery in any case, which is logically discounted via risk-free rate and a risky part which is the excess over recovery and discounted by risk-neutral probability

[24] even though $N(DD) = N(d_2)$ is definitely a good indicator of firm's PD in fact the DPT is not the same as we assume in our calculation. Therefore we can not use $N(DD) = N(d_2)$ as an ultimate relevant indicator of PD but we need to search for empirical EDF based on DD.

[25] firms are usually classified with respect to economic sectors

[26] logically lower DD implies higher EDF

[27] which is usually the case

PD_{rn}. The present value PV of bond discounted in accordance with the outlined method is formally expressed as (Witzany, 2010)

$$PV = \sum \text{riskfree CF} + \sum \text{risky CF}, \tag{4.14}$$

and more precisely as

$$PV = \sum_{i=1}^{n} \overbrace{CF_{t_i} e^{-rt_i}(1 - LGD)}^{\text{risk-free part}} + \sum_{i=1}^{n} \overbrace{CF_{t_i} e^{-rt_i}(1 - PD_{rn}(t_i))LGD}^{\text{risky part}}, \tag{4.15}$$

under the assumption there are $i = 1, 2, \ldots, n$ discounting periods.

Having derived the way how to discount a bond we can ignite a portfolio simulation.

4.4 KMV and portfolio simulation

The crucial factor influencing the performance of portfolio is **correlation**. We can apply very similar approach in KMV to the one exercised in CreditMetrics. Since the randomness and uncertainty affect the portfolio via path of asset walk the correlation must be incorporated into the lognormal property[28] to provide correlated paths. Hence we similarly the Section 2.4 decompose the correlation matrix into $\mathbf{L}$, $\mathbf{U}$ matrices using the *Cholesky decomposition* and subsequently we generate a random vector $\boldsymbol{\theta} = (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(J)})^T$, where $\theta^{(j)} \sim N(0, 1)$ for debtors $j = 1, 2, 3, \ldots, J$. Correlated values $\tilde{\theta}^{(j)}$ are calculated as in (2.17)

$$\tilde{\boldsymbol{\theta}} = \mathbf{A}\boldsymbol{\theta}. \tag{4.16}$$

Elements $\tilde{\theta}^{(j)}$ of correlated values'-vector $\tilde{\boldsymbol{\theta}}$ are then used when sampling over the lognormal property[29]. Thus, we manage to simulate M scenarios for time T. Contrary to CreditMetrics the outputs are not rating grades (i.e. which would mean direct determination of bond prices) but asset values at time T that over DD substitute the rating grades. The subsequent problem is therefore to determine the value of a bond portfolio at time T with respect to $A(T)$. Hence we follow the formula (4.15) with a time shift to time T, more precisely, we are interested in PV_T. What needs to be revaluated for the PV_T are risk-neutral probabilities for the following time horizon, since they are implied by the the

[28] or the discrete-path modeling

[29] when we sample we firstly have to translate $\tilde{\theta}^{(j)} \sim N(0, 1)$ back to probability that is determinative for the sampling

asset value that is modeled (i.e. $A(T)$)[30]. For determining PV_T we therefore need to know only $A(T)$ which is the simulated one and which implies also future values of risk-neutral probability. However, the calculation also involves forward risk-free rate for discounting, which can be calculated using the same yield curve decomposition as in (2.2) or we might use *yield curve interpolation*[31]. Sum of PV_T over all obligors then constitutes the total expected value of a bond portfolio at time horizon T. We can mathematically formulate the total portfolio value in scenario m, for $m = 1, 2, 3, \ldots, M$ as

$$V_{(m)} = \sum_{j=1}^{J} PV_T^{(j)}, \tag{4.17}$$

where $V(m)$ is the portfolio value of in m-th simulated scenario, $PV_T^{(j)}$ is the "present" value of j-obligor's bond at time T. The output of M simulations is subsequently visualized and evaluated in the same principle as in the case of CreditMetrics. It needs to be noted that we may also adjust the DPT level when we are deriving the risk-neutral probabilities for future periods. However, such a step requires really an expert-based assessment since there is no clear rule or rather a scheme how the DPT changes in time. Some approaches also suggest to simulate the DPT values but such a measurement need not provide better results, it only makes the output more complex.

[30] the asset volatility σ_A is assumed to be constant throughout the entire observed period

[31] we will exercise such an approach in our modeling part related to KMV

5 Summary of the theoretical part

In the previous chapters we have theoretically described three commonly used credit risk portfolio models. The characterization should also provide a comprehensive tutorial how to perform modeling in a framework of those models. We also may have discovered some weaknesses of the selected models and however also their strengths mostly induced by assumptions they rely on.

Generally said, from the outlined models **CreditRisk$^+$** is the most simple one, largely owing to the fact it **does not try to capture any real-world risk dynamics** but only uses rigid default rates with a variability factor of standard deviation. CreditRisk$^+$ thus relies clearly on a power of probability which usually proves to be accurate on numerous samples. Since there is no real module how to incorporate correlations into CreditRisk$^+$ the model only uses a very simplified interface named sector analysis which tries to compensate the missing correlation extension. In spite of all the simplifications the model performs well on large portfolios, usually of retail clients. The probability theory can essentially work well in environment where is high common *homogeneity* and *low correlation* among subjects. As it also does not use any simulations the model is very fast and provides stable results. However, it does not fit for portfolios that need to be evaluated with some expert-based systems scrutinizing the causes of default rather than utilizing the stand-alone plain probability of default.

CreditMetrics was another discussed model. It is more complex and more robust than CreditRisk$^+$. The modeling in CreditMetrics involves **rating-based** bond discounting for which there is a causality with Merton model approach. On the other hand, the role of Merton model is only subtle in CreditMetrics, it does not cover any simulations of asset walk. CreditMetrics model's environment allows us to **incorporate correlations** for debtors which results into correlated simulated bond ratings. After the possible future ratings are set up, bonds can be valuated and the value of portfolio is thus calculated as a sum of all partial bond values. Finally, distribution of values is generated and it nicely demonstrates the possible outcomes. Employment of CreditMetrics is high when we **evaluate non-homogenous credit portfolio** and also when the debtor ratings are estimated accurately by the rating agencies. The positive is that the model does not require any explicit firm-related capital market data since it argues the data are incorporated in the ratings and thus it makes the computation easier. To the contrary, the amount of data

needed to construct well working model is large since we also need term structure of yield curves. To the addition the model can be relatively easily upgraded with some extensions (we will introduce some of them in the applied part of this thesis).

Last of all we were going through the **KMV model**. It also utilizes the Merton model engine but in a more rigorous way because in KMV the **asset walk itself is directly modeled**. Option pricing theory is an important constituent of this model, we consider firm's equity value as a call option premium, for which the strike price is the firm's default point. For the modeling itself it is necessary to determine firstly the asset value and volatility and subsequently we can model random asset walks which represent possible future states of the company. Company's riskiness is expressed by two types of probabilities, firstly the **real-world probability** EDF and secondly by so called **risk-neutral probability**. Real-world probability expresses the true probability of default of the company, whereas risk-neutral probability is used for discounting and therefore it is an important input for modeling of bond values scenarios. Simulating the asset walks we obtain distribution of asset value at a future time horizon that determines both of the probabilities and hence also future values of a bond. The model also allows to incorporate a correlation of debtors, which is integrated into the model via correlated asset walk. KMV is a perfect model for assessing riskiness of debtors with well accessible capital market data because it can provide very market-sensitive result. The negative is **it cannot be used for companies whose capital market data are not directly observable**.

Our intention is not only to provide theoretical description of the models but also their practical demonstration. Therefore the next part of this thesis is devoted to a demonstration on a hypothetical credit portfolio to verify the performance of selected models.

Part II

Applied modeling

6 Credit portfolio

6.1 Portfolio selection problem

For our credit portfolio analysis, it is firstly necessary to define a reasonable debt portfolio on which the performance of selected models will be tested. Since the models are substantially different concerning their suitability for the size of portfolio[32] and moreover differ also in the appropriateness for large subjects and/or retail-based portfolios we have to accept a compromise if we want to use only one portfolio for all the models. Hence, to enable us to get all the necessary data we need a portfolio of bonds, whose issuers are quoted on a stock market and simultaneously for which the ratings are also accessible. Such a portfolio can be preferably created from *US stock market* data since the US stock market is well developed and all the necessary data are at disposal. As a next point, we need to make a choice of the portfolio size. We reckon that in a framework of all the models, when taking into account the inseparable "trade-off", a portfolio of 20 debtors is appropriate. As a last point, we have to choose specific companies (bond issuers) and also to determine notional attributes for their bonds. Because we do *not* want the portfolio to be *trivial* (homogenous) we will pick firms of many different rating grades and also variously sensitive to the market activity. All of the above symbolizes that the portfolio in our artificial "lab environment" must be chosen with care to allow the models demonstrate how they operate and to minimize the trade-offs that unfortunately need to be taken and which decrease the strengths of each model.

[32] in other words to the number of claims, not the absolute value of obligations

6.2 The portfolio

Since we want our study to be *as close to reality as possible*, we have picked a portfolio comprising 20 bonds of issuers throughout various sectors in USA. Moreover, the bonds are of substantially different **maturities** (**2011** up to **2016**) and of course also their **quality is inhomogeneous**. Most of them pay their **coupons semi-annually** but there is one bond paying the coupon quarterly. The bonds were rated by both *Moody's* and *Fitch* which eventually eased getting complete information about the bonds. Generally, the bonds were firstly chosen from **Yahoo! Finance**, where overall rough data are listed and subsequently more precisely we analyzed concrete bond issues in the databases of rating agencies. We intentionally picked bonds of rather **speculative grade** to demonstrate the riskiness of such a portfolio. Exactly, there are 13 bonds of speculative grade (also called "*high yield*" or "*below-investment*") and the rest 7 bonds are of an investment grade. Note that in spite of an investment grade, there are no claims rated *Aaa* or *Aa* therefore even the investment grade is not "that much" secure. Regarding the *Loss Given Default* (LGD) we use data for a specific bond issue, in the case such data were published. If not, we use weighted mean *LGD* for North American companies in 2009, published by Moody's. Last assumption that needs to be taken is the **time horizon**. Precisely we firstly need to fix a "*now*" date and further we fix a time-shifted "*future*" date. With respect to gathered data we decided to use **1.6.2010** as the "**current**" date and the one year later data **1.6.2011** as the **future date**. This is very important because the reader must always bear in mind the assumed time horizons as they influence the whole portfolio calculation and moreover the data entering the mode are pegged to those dates. Finally see the selected portfolio in Table 1. Note that this table is simplified to fit into a page. Coupon of debtor no. 19 is marked by asterisk '*' since it pays coupons quarterly whereas the other debtors semiannually.

However, there are some necessary simplifications to be taken when we want to analyze the portfolio within a model. We will go through these simplifications in a chapter of every model since they are different. Firstly, we go through CreditMetrics.

#	Company name	Sector	Rating	LGD	Par ($ mm)	Coupon	Maturity
1	Tenet Healthcare	Healthcare	Caa1	85%	10.5	7.38%	1.2.2013
2	Rite Aid	Retail	Ca	95%	5.0	6.88%	15.8.2013
3	Sprint Nextel	Teleco	Ba2	74%	7.0	6.00%	1.12.2016
4	Cablevision	Media	B1	90%	8.5	8.00%	15.4.2012
5	Commercial metals	Mining	Baa2	65%	14.0	5.63%	15.11.2013
6	Dynegy	Energy	Caa1	97%	6.0	8.75%	15.2.2012
7	Ferro Corporation	Chemical	B1	71%	20.5	6.50%	15.8.2013
8	Interpublic Group	Media & M.	Ba2	52%	3.0	6.25%	15.11.2014
9	Iconix Brand	Manufacture	B1	72%	6.0	1.88%	30.6.2012
10	Saks Incorporated	Retail	B2	54%	5.5	9.88%	1.10.2011
11	Standard Pacific	Real-Estate	B3	48%	11.0	6.25%	1.4.2014
12	Aon Corporation	Insurance	Baa2	65%	7.0	7.38%	14.12.2012
13	Avnet	Technology	Baa3	65%	6.0	6.00%	1.9.2015
14	Comerica Bank	Banking	A1	69%	9.0	5.75%	21.11.2016
15	Costco Wholesale C.	Retail	A2	65%	4.5	5.30%	15.6.2012
16	Weatherford Int.	Energy	Baa1	65%	5.0	5.15%	15.3.2013
17	Eastman Kodak C.	Technology	Caa1	67%	7.0	7.25%	15.11.2013
18	American Axle	Automotive	Caa2	76%	8.0	5.25%	11.2.2014
19	Continental Airlines	Airlines	Ba1	56%	13.5	6.56%*	15.2.2012
20	Hovnanian Enterprises	Real-Estate	Caa1	95%	7.0	7.50%	15.5.2016

Table 1: *Selected portfolio*

7 Applied CreditMetrics

Firstly, respecting the outline of the theoretical part of this thesis, we would like to turn our attention to CreditMetrics. Modeling in CreditMetrics requires perfectly prepared consistent large amount of data[33] to show its best performance. Hence, our first intention is to collect and – if it is inevitable – also to adjust the data. When the data is prepared, we are enabled to run the simulation and analyze the results.

7.1 Data procurement and customization

7.1.1 Company bonds yield curves

Since the core of CreditMetrics is **risk-adjusted discounting**, we firstly need to obtain the yield curves related to a bond quality and maturity. The data for bond yields we collected from Reuters were quoted as spreads over a risk-free rate, hence we as the very first data need term structure of US Treasuries. Having the US Treasuries, we can easily derive time structure of company bonds. But there are some difficulties. In our portfolio we have bonds with maturities ranging from 1 up to almost 7 years and we know enumerative US Treasuries' yields for maturities (in our case, relaxed from extremely short or long maturities) for maturities in 1, 2, 3, 5 and 7 years. Hence we need to approximate yields for 4 and 6 years maturities. Those maturities were therefore derived by approximating the yield curve by interpolation. And the same measurement was later performed also for company bond yield curves. Subsequently we have a structure of bond yields as shown in Figure 4. We can see the time structure of Aaa, Aa, A, and even Baa bonds is relative close to US Treasury bonds in short-time-horizon maturities. The landmarks are Ba bonds, for which the curve is substantially distant from the investment grade bonds. Note that the yields are in general very low since the US Treasury bonds yields are slight as *FED* had decreased the interest rate and bond are thus also affected.

Another problem is implied by the pattern of real-world bonds portfolio in our case because the bonds do not have maturities exactly at integer time but in variably between two dates in future. Hence we need some mechanism to determine rates for discounting related to a point $T^{(maturity)} \notin \mathbb{Z}^+$. It will be closer discussed in later chapters.

[33] especially from rating agencies

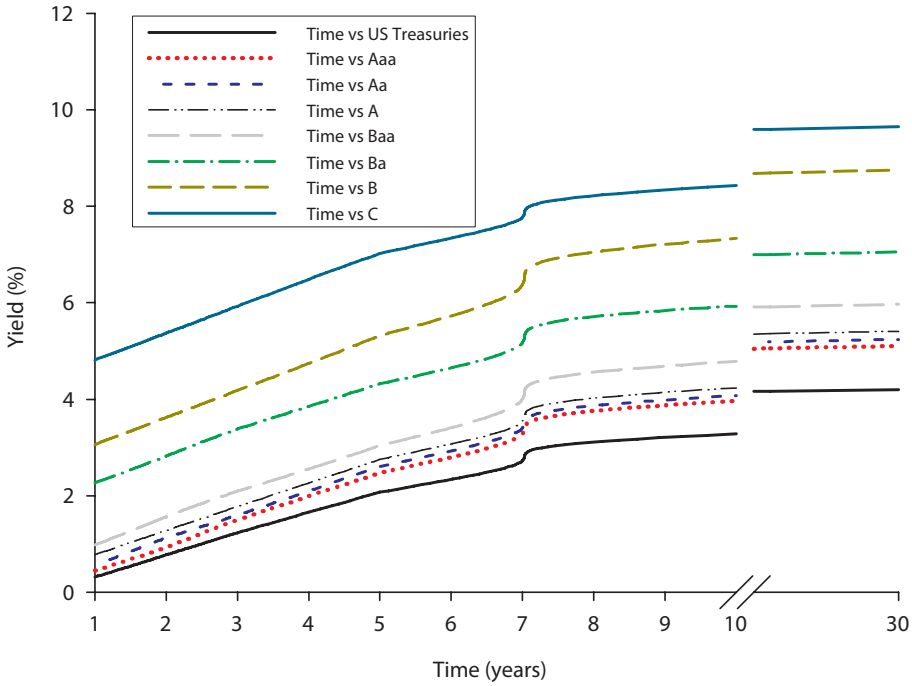

Figure 4: *Bond yields*

7.1.2 Transition matrix adjustments

Since we use ratings of Moody's for bond valuation it follows that we also have to use Moody's transition matrix to make the calculation consistent. What is more, Moody's divides transition matrices into "North America"– and "Europe"–region which enables us to use more accurate data in case of USA companies with North American matrix selection. For the reason the term-structure of yields is divided into 7 categories we require transition matrix with the same quality. Such a transition matrix is published, nonetheless we encounter another problem – namely the *withdrawn ratings* column is extra and consequently the matrix is not of a square type.

Let us take a look firstly at the initial transition matrix published by Moody's as show in Table 2. We now have a matrix $I \times J = 7 \times 9$ which is a problem for as it is not of a squared type. Firstly, to make a square matrix, we need to add a row "Defaults" under the C row and in addition we also have to make one more rearrangement to eliminate the WR column. Hence, following instructions from (Bluhm et al., 2010) we dissolve the WR column into classical rating categories using a rule of category weight. More precisely we

From/To	Aaa	Aa	A	Baa	Ba	B	C	Default	WR
Aaa	87.77%	7.47%	0.32%	0.00%	0.00%	0.00%	0.00%	0.00%	4.44%
Aa	1.05%	84.53%	8.43%	0.35%	0.06%	0.04%	0.01%	0.01%	5.51%
A	0.06%	2.15%	86.47%	5.81%	0.58%	0.15%	0.05%	0.04%	4.69%
Baa	0.06%	0.21%	3.78%	84.59%	4.39%	1.01%	0.23%	0.22%	5.51%
Ba	0.01%	0.07%	0.41%	5.20%	73.66%	8.32%	0.62%	1.28%	10.42%
B	0.01%	0.04%	0.13%	0.36%	4.09%	73.98%	6.52%	4.45%	10.40%
C	0.00%	0.00%	0.02%	0.17%	0.37%	7.17%	63.57%	16.46%	12.24%

Table 2: *One-year transition matrix (unmodified)*

sum up all the values in a row excluding WR and based on the weights given by classical rating values we dissolve the WR category.

Mathematically for each row $i = 1, 2, \ldots, 7$ we define a robustness value

$$robustness_i = \sum_{j=1}^{8} V_{i,j}, \tag{7.1}$$

as a sum of transition matrix values $V_{i,j}$, which is the same as

$$robustness_i = 1 - WR_i. \tag{7.2}$$

For instance, for row $i = 1$ in our case the robustness would be $1 - 0,044 = 0.956$. Then we calculate weighted new value of every rating category in i-row. It gives

$$V_{i,j}^{(new)} = \frac{V_{i,j}}{robustness_i}. \tag{7.3}$$

For our first case the $V_{1,1}^{(new)}$ would be $V_{1,1}^{(new)} = 0.8777/0.956 = 0.9181$. As a check up we must get sum of all values $V_{i,j}^{(new)}$ in every i-row equalling 1. Logically the total increment for each value in a row i must give WR_i.

After the rearrangement we obtain a new modified squared transition matrix as shown in Table 3. Exactly this table will be used in our calculations.

7.1.3 Debtor rating aggregation

One more necessary compromise we must make is to merge some of debtor's ratings into more general groups. Otherwise it would be impossible to discount such debtors as there

From/To	Aaa	Aa	A	Baa	Ba	B	C	Default
Aaa	91.85%	7.81%	0.34%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa	1.11%	89.46%	8.92%	0.37%	0.06%	0.05%	0.02%	0.02%
A	0.06%	2.26%	90.72%	6.09%	0.60%	0.16%	0.05%	0.04%
Baa	0.06%	0.22%	4.00%	89.53%	4.64%	1.06%	0.24%	0.23%
Ba	0.01%	0.08%	0.45%	5.81%	82.23%	9.28%	0.69%	1.43%
B	0.01%	0.05%	0.15%	0.41%	4.57%	82.57%	7.28%	4.97%
C	0.00%	0.00%	0.02%	0.19%	0.43%	8.17%	72.43%	18.75%
Default	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	100.00%

Table 3: *One-year transition matrix (modified)*

are only discount curves for more general ratings. Since we have very exactly rated bonds such as *Baa2, Baa1* we merge close ratings into one of 7 general categories shown in modified rating migration matrix.

7.1.4 Correlation matrix

Last but not least we must determine the correlation structure among debtors. It is very important since it implies severity of "*fat tail*". If the correlation is low, the portfolio is diversified and does not tend to attain extreme values, contrary to a heavily correlated portfolio. We proceed from the assumption the economic crisis is still persisting, which implies the debtors are fragile. To personify and incorporate such a situation by correlations we estimate the correlation structure among debtors on last 2 years time series data, when the crisis essentially started. The correlation is computed from *weekly logarithmic returns* on debtors share prices. However, even the correlation matrix had to undergo adjustments owing to p-values in correlations matrix. If $H_0 : \rho_{i,j} = 0$, we need the p-value to be 0.05 at maximum to reject H_0 and claim there is a statistically significant correlation between debtors. Computer program gave us a correlation structure including p-values, so we then manually filtered out the insignificant correlations and replaced them by 0. The correlations are in general very intensive. It just underscores what we expected it to be it like.

Having all necessary components for the modeling, we can move to the next part dedicated directly to modeling itself.

7.2 Determination of the present value of debt

Firstly, to be able to discount the bonds, we must resolve the *non-integer value of time horizon problem* we had already mentioned in Section 7.1.1. Hence in our CreditMetrics model we apply a very simple but effective mechanism to estimate interim values of yield curves between two time horizons τ_1 and τ_2, where $\tau_1 < \tau_2$ and $\tau_1, \tau_2 \in \mathbb{Z}^+$. Let us define $Y(0, \tau)$ to be yield for a period from now till time $\tau > 0, \tau \in \mathbb{R}^+$. It is obvious from the pattern of observed yield curves that

$$Y(0, \tau_1) < Y(0, \tau) < Y(0, \tau_2) \Leftrightarrow \tau_1 < \tau < \tau_2. \tag{7.4}$$

Hence we may approximate the value of $Y(0, \tau)$, for $\tau_1 < \tau < \tau_2$, if we know $Y(0, \tau_1)$ and $Y(0, \tau_2)$. Since the trend is upward sloping and higher time gives higher yields, we may use simple linear approximation. However, there is an error, deviation from real-world but generally even such an approximation serves well for our purposes. The only necessary assumption is that we can never use the approximation for longer time horizons, the error would be then substantial and our calculation biased.

Let therefore for $\tau_1 < \tau < \tau_2$

$$Y\left(0, \tau\right) = a\tau + b, \tag{7.5}$$

where a and b are the unknown coefficients. It is straightforward to use 2 equations

$$Y(0, \tau_1) = a\tau_1 + b \text{ and } Y(0, \tau_2) = a\tau_2 + b. \tag{7.6}$$

Since we know $Y(0, \tau_1)$ and $Y(0, \tau_2)$, the a and b coefficients can be expressed as

$$a = \frac{Y(0, \tau_2) - Y(0, \tau_1)}{\tau_2 - \tau_1} \text{ and } b = Y(0, \tau_1) - a\tau_1. \tag{7.7}$$

When we further fulfil the assumption of only one year difference, the expression $\tau_1 - \tau_2 = 1$ implies

$$a = Y(0, \tau_2) - Y(0, \tau_1) \text{ and } b = Y(0, \tau_1) - a\tau_1, \tag{7.8}$$

which is a nice simplification. For example if $Y(0, 2)$ and $Y(0, 3)$ for Aaa-rated bond is 0.93 and 1.50 respectively and we want $Y(0, \tau)$, where $\tau = 2.8$, then $Y(0, 2.8) = 1.386$ and it holds that $0.93 < 1.386 < 1.50$.

Hence, knowing the discounting mechanics, we can determine the present value of our portfolio. We intentionally relax the assumption the bonds are about to be bought, since

we assume we already have them in the debt portfolio. It means there is nothing such as *aliquot settlement* to counterparty etc. assumed. The bond discounting is then trivial and does not need closer attention. The present value of the claims is calculated to be **$173.31 million**.

It is higher than the **par value** of debt which is **$164 million** but we have to take into account that the coupons are relatively generous, discounting rates low due to low treasury basement-level increased of the credit spread and the discounting also assumes the ratings do not change.

7.3 CreditMetrics kernel

Now, we are focusing on the future states of the world. More precisely, we are interested in the portfolio value at time $T = 1$ (date 1.6.2011). Such estimation however requires a simulation. Hence, we firstly need to create the **migration thresholds** to enable us simulate rating transition. Based on the threshold-creation algorithm stated in Section 2.3 we created from our modified transition matrix following matrix of thresholds depicted in Table 4. The interpretation of the values is very trivial. For instance if have a *Baa* rated

From/To	Aaa	Aa	A	Baa	Ba	B	C	Default
Aaa		-1.395	-2.709					
Aa		2.286	-1.315	-2.567	-2.991	-3.164	-3.427	-3.611
A		3.228	1.992	-1.479	-2.381	-2.794	-3.096	-3.328
Baa		3.224	2.764	1.718	-1.540	-2.160	-2.595	-2.828
Ba		3.671	3.106	2.543	1.525	-1.205	-2.028	-2.188
B		3.622	3.221	2.863	2.502	1.627	-1.163	-1.648
C				3.497	2.851	2.487	1.352	-0.887

Table 4: *Calculated migration thresholds* Z_i

bond and we sample $z \sim N(0,1)$ value e.g. $2.764 < z < 3.224$, new rating is assigned *Aa*. If we breach the upper threshold of *Baa* for *Aa* e.g. $z > 3.224$ then *Baa* turns to *Aaa*. Conversely, if for the same bond we sample $z < -2.188$, the debtor defaults.

However, if we sampled from $N(0,1)$ it would mean the debtors migrations are uncorrelated which is contrast with reality depicted in our correlation matrix. Hence we need to transform the correlation structure into the randomly generated z numbers. It is done via lower triangular *Cholesky matrix* $\mathbf{A}$ that we obtain from our correlation matrix $\mathbf{\Sigma}$ by

its decomposition. It is precisely described in the Sections 2.4 and 2.5.

After multiplying random vector $\boldsymbol{\theta}$ by lower triangular matrix $\mathbf{A}$ from the left side, we obtain correlated vector $\tilde{\boldsymbol{\theta}} = \mathbf{A}\boldsymbol{\theta}$ which contains the correlated $z^{(j)*}$ values for rating migration process. With respect to the precision, we have **generated 50 000 scenarios** containing $z^{(j)*}$ values. These values are then simply evaluated according to thresholds and based on the initial rating of the j-debtor the model assigns in m-scenario new rating $s_m(z^{(j)*})$.

In the last step we basically completed the portfolio simulation, what remains open is the evaluation of every of the newly created $m = 1, 2, 3, \ldots, M$ scenario. By running 50 000 simulations, we have totally generated 1 000 000 of new ratings $s_m(z^{(j)*})$. Now the bonds with respect the newly assigned ratings are to be evaluated. But, again, analogically Section 7.2 we need forward yield curves to do so. Forward yield curves are simply derived applying formula (2.2). And again, the discount curves interims are approximated by linear approximation stated in Section 7.2. What has remained unsolved are the coupon payments. We have therefore developed an extension to the basic problem to handle it.

7.4 Own extensions to the plain model

Since the plain model does not capture some of very important economic occurrences, we have implemented several own extensions in it. Some of them are important for the true value of the portfolio, the others are rather goodies, accessing extra features but giving no better portfolio value estimation. We begin chronologically with the most important one.

7.4.1 Time-adaptive conditional coupon payments until default module

We want the model to be as close to reality as possible which involves reasonable approach to evaluating paid/unpaid coupons within time horizon $T = 1$. The plain CreditMetrics does not tackle this problem at all but we do. Suppose every debtor obliged to pay the coupon semi-annually either defaults or not within the time horizon given by a closing time $T = 1$. If there is no default by $T = 1$, it is clear the coupons have been paid. On the other hand, if we generate **Default state** by the time $T = 1$ it does not say us anything about "when exactly" during $0 < t \leq 1$ the debtor defaulted implying uncertainty about coupon payments. The debtor might have defaulted within an infinitesimally small time period after our "present" date and paid no coupons or he might have defaulted

infinitesimally close to the time horizon $T = 1$ and definitely paid both of his coupons. To capture these occurrences, we implemented into the model a mechanism that exerts to simulate randomness of the debtor's coupon payments if he eventually defaulted. It would be however *unfair* to give the same probability of the coupon payments to every debtor since their **coupon payment intervals are different**. That brought us to simulate the probability of paid coupon relative to the time remaining to coupon payment.

Consider an illustrative example. We have a debtor X and a debtor Y, whose coupon payments are realized at time $T_{x,1} = 0.1$, $T_{x,2} = 0.6$ and $T_{y,1} = 0.45$ and $T_{y,2} = 0.95$. If we generate defaults at $T = 1$ for both debtors, then how should we model their coupon payments until default? We firstly relax the influence of the debtors' X, Y initial ratings because it is no object if both of them are defaulted at time $T = 1$. If we address the same probability of paying the coupons to both debtors it would be unfair because it is obvious that the X debtor is much more likely to pay the 1st coupon than the Y debtor since X must survive substantially smaller time interval to be able to pay the 1st coupon. Hence in this case we apply model as follows.

If we at $t = 0$ model default by time $T = 1$, it automatically triggers another random procedure for modeling the coupon payments until default. We use the relative portion of time necessary to survive as a threshold for a random number $r \in < 0; 1 >$. The probability of the payment of the 1st coupon of the X debtor is relative to time to the payment $\Pr[X \text{ pays 1st coupon}] = 1 - T_{x,1} = 0.9$, in modeling the random $r \in < 0; 1 >$ must be sampled $r < 0.9$ to conclude the 1st coupon payment occurred. The Y debtor is less probable to pay the 1st coupon than X since there is a longer period needed to survive to the 1st payment. It means for the Y debtor $\Pr[Y \text{ pays 1st coupon}] = 1 - T_{y,1} = 0.55$ and for the survival his random r must be < 0.55.

But however, there are arranged **2 coupon payments** during the year since the coupons are paid semi-annually. The problem is that if we just simply attributed intersection of both time intervals as the probability of 2nd coupon payment, it might mathematically occur that the debtor pays coupon 2 and no coupon 1, which is in reality a nonsense. Therefore we implemented "*admitting condition*" that allows to sample the second payment if and only if the 1st payment was successful. If the 1st toss was unsuccessful (i.e. no 1st coupon was paid), it automatically also rejects any further toss for coupon 2. It also follows that the second interval must be rescaled and the proba-

bility of the second coupon payment must be adjusted. Hence for X, if he survived to $T_{x,1} = 0.1$, he must survive another 0.5 time units to pay the second coupon. But the interval is rescaled because we do not take into account the past 0.1 time units anymore. Hence the tossing for X and coupon 2 must give following probabilities characteristics. $1 - 0.1 = 0.9$ is the remaining time to $T = 1$ and 0.5 time units left to $T_{x,2}$ the probability of survival (i.e. payment of the 2nd coupon) with respect to remaining time interval is $\Pr[X \text{ pays 2nd coupon}|\text{1st c. paid}] = (1 - 0.1 - 0.5)/(1 - 0.1) = 0.44$ and the obligor must toss $r < 0.44$ to pay also the second coupon. For the Y debtor it is as follows. First coupon payment probability has already been calculated as 0.55, it must be sampled $r < 0.55$ (or alternatively $r \geq 0.45$) and for the second coupon the conditional probability is $\Pr[Y \text{ pays 2nd coupon}|\text{1st c. was paid}] = (1 - 0.45 - 0.5)/(1 - 0.45) \cong 0.091$ hence sampled $r < 0.091$ to allow Y to pay the second coupon.

The same approach can be used for quarterly paid coupons. The extension with random coupons is very important for the final portfolio value since the accrued coupon payments substantially change the portfolio value.

7.4.2 Risk-free rate adjustment module

To make it even more robust, we implemented also one more purely economic assumption. If the coupon is paid we cannot suppose the bank will be *phlegmatic* with the earned coupon. Hence the model assumes the obtained coupon earns a risk-free rate relative to the time period for which it holds till time $T = 1$[34].

7.4.3 Correlation adjuster

Another extra extension is so called "*Correlation adjuster*" which is a controller that simply decreases or increases the correlation intensity throughout the whole portfolio and one can finally observe the impact of correlation on the portfolio performance. The correlation adjuster is a powerful feature of our model, implemented into attached worksheet.

[34] Even though the one year risk free rate is currently 0.3% p.a. it would be economically hardly justifiable to assume no earnings whatever minor they are. It rather conforms the general economic idea but for the calculation it is quite negligible.

7.4.4 Track-back module

We also implemented into the model a "track-back" function so that it is possible to browse retrospectively through all the generated scenarios and observe what coupons which obligor had paid, what was his rating in m-th scenario etc. It is also possible to fill the cells by generated ratings so that we in a well arranged way may observe individual modeled scenarios.

7.5 Portfolio simulation results, assessment and summary

Having mentioned the methodology and adjustments that entered the simulation we can now analyze the results of final simulation. We have simulated **50 000 scenarios** to ensure convincing and reliable results. Table 5 and Figure 5 summarize main outputs of the simulation.

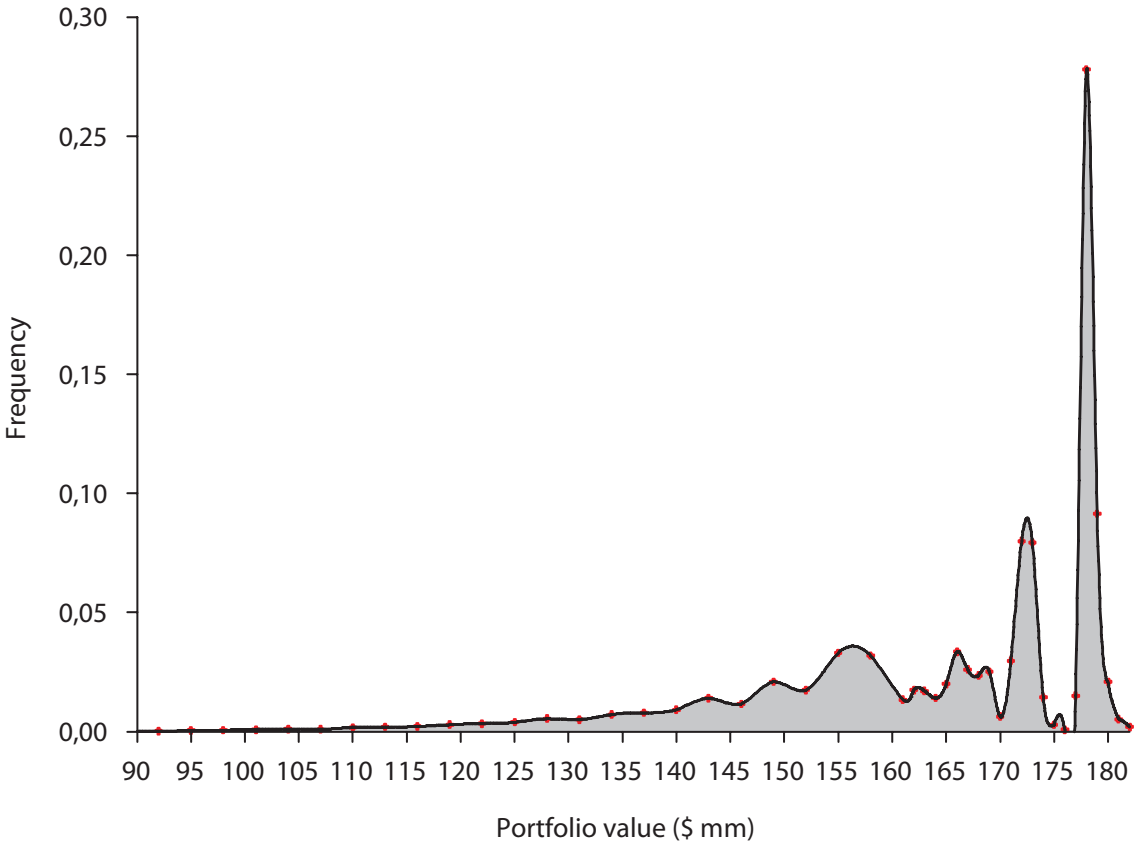

Figure 5: *CreditMetrics portfolio value simulation*

On the distribution curve, we may observe logarithmic pattern of credit returns. The

Summary statistics ($ mm)			
Runs	**Mean value**	**Standard deviation**	**Unexpected loss**
50 000	169.14	13.31	45.93
Mode	**Median**	**1% quantile**	**99% quantile**
178.50	173.20	118.07	180.76
Worst case scenario		**Best case scenario**	
66.95		184.34	
Average number of defaults in every run			
1.43			

Table 5: *CreditMetrics simulation results*

most returns are cumulated in the area around $179 mm and around $172 mm. On the other hand, there is a long "fat tail" causing severe losses if such an scenario occurs. There are also some peculiarities in our case since there is a remarkable "slump" of probability just around value $176.5 mm. It is caused especially by the fact that the portfolio is still relatively small and if there is at least one default in the scenario we are moved to area with values up to $174 mm. Also important factor is that the bond nominal values are substantially variable causing "moulding" on the curve. The portfolio is very sensitive to high claim values therefore the granulation occurs.

As the very last statistic related to CreditMetrics we introduce a chart depicting the distribution of ratings to compare. Figure 6 demonstrates ratings distribution before we ignited the simulation and compares it with newly created rating distribution that originated from Monte Carlo simulation.

We can notice that the most "active" rating group were debtors C who substantially migrated and many of them have eventually defaulted. On the other hand, there are also some minor occurrences of Aaa (0.02%) and Aa (0.31%) ratings.

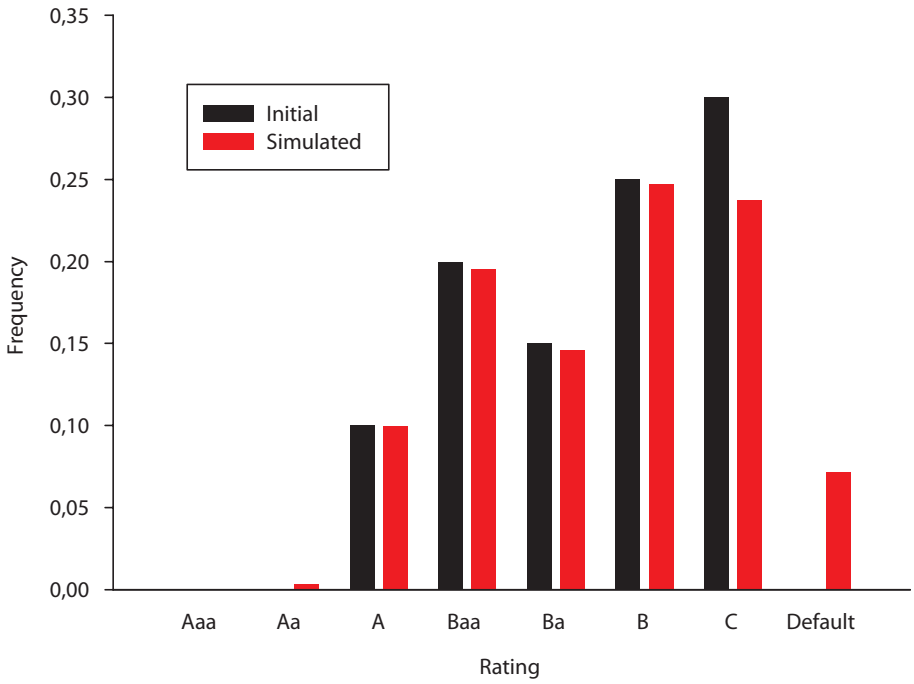

Figure 6: *CreditMetrics: rating distribution before and after simulation*

8 Applied CreditRisk$^+$

In this chapter we are going to apply model **CreditRisk$^+$**. As it has already been outlined in the theoretical part (see Section 3), CreditRisk$^+$ is primarily not intended to be used for small portfolios consisting of company debts. Hence it is not even easy to adjust such a model for a company bonds portfolio. Due to very restrictive but necessary assumptions also the results might be biased more than one would expect. On the other hand, the model at least offers several approaches that can give different results and the analyst does not necessarily need to rely on one result as in the case of CreditMetrics. As the main drawback of this model we however consider the absence of economically justifiable approach to correlations and also some particularities implied by the qualities of *Poisson distribution*. Since the incidence of the restrictions implied by the CreditRisk$^+$ makes it rather difficult to apply it to a company debt portfolio, the following analysis will be more likely an illustrative example but the results can not be treated as very relevant. We will firstly go through simple sector analysis with no volatilities in default rates and also the portfolio is treated as one large sector. Then, more advanced approach will follow where the plain model will be upgraded for default volatilities. Finally, the most advanced approach taking into account several sectors with default rate volatilities will be demonstrated.

8.1 Data preparation and modeling preliminaries

What we need to run CreditRisk$^+$ at least in the basic version, are **probabilities of default** and the **exposures**. To follow our consistency-preservation principle, we use the same default probabilities as in the matrix given in the part about CreditMetrics. The exposures are trickier. The problem is CreditRisk$^+$ does *not* analyze "which obligor" defaulted. It just returns an overall portfolio number which is given by convolutions of loss-probability generating functions. Since it is impossible to determine which obligor defaulted, it is also impossible to map recovery rates to modeled defaults. Moreover the essential model's output is a distribution of losses not a distribution portfolio value as in the case of CreditMetrics. Therefore we can not contrary CreditMetrics inject the exposure attribute for each obligor during the modeling process but we must apply a different approach. It seems logical to prepare the data so that the model does not even need to take the RR or complementary LGD into account. Therefore we simply separate the real nominal exposures to risky (LGD) part and a secure part (represented by RR). Hence the final exposure for each obligor is only the LGD part. If we want to perform portfolio value benchmarking with other models, we just simply decrease the nominal values of the predicted losses.

In addition there is one more extra serious problem afterwards, namely the coupons. In CreditMetrics/KMV, we can more or less with difficulties but still incorporate the coupons payments. See e.g. our extension which nicely captures and simulates the (un)paid coupons within default in the Section 7.4.1. But again, in CreditRisk$^+$ the ex-post identities of defaulted debtors are *de facto* unknown and hence any "coupon until default" framework is unthinkable. The main issue for model comparison is that the amount of paid/unpaid coupons is definitely not negligible[35]. Therefore, we can not take the coupons payments into account in CreditRisk$^+$. If wanted to do so, we could only apply very trivial method of dissolving coupons payments within $T = 1$ according to the loss distribution[36]. But we could only do it under an assumption of linear relation of coupon payments and losses (or defaults). Such approach is, however, logical but hardly mathematically justifiable and therefore we must relax the coupon payments altogether.

To start model the risk, we firstly prepare the **exposure bands**. When we want to do

[35] the ultimate difference of paid/unpaid coupons makes approximately \$10.6 mm

[36] logically higher losses would be implied from higher frequencies of default and hence the the accrued coupons payments will be assumed lower and vice versa

so, we immediately encounter another problem – bands accuracy. To give some reasons, suppose the following. Assume the size of band as $B = \$0.05$ mm, which wouldn't mean severe rounding error from actual numbers if we expressed every debtors' exposure in terms of B. The reason why we can not to do so is that we would get very large numbers of bands m (in fact much higher than the number of debtors in our portfolio, precisely 291) so that the band approach would lose its purpose as there would be no grouping. Therefore, we must rescale the B to a substantially higher number. To be able to employ the nice idea of band-approach to the portfolio, we must accept $B = \$1.0$ mm which unfortunately causes *significant error* from actual number when we have to use the rounding to get integers. But at least we can then demonstrate the nice idea of band approach. Remind we have a simplified portfolio as shown in Table 6.

Debtor	**Exposure**	**PD**	**Debtor**	**Exposure**	**PD**
1	8.93	0.19	11	5.28	0.05
2	4.75	0.19	12	4.53	0.00
3	5.18	0.01	13	3.88	0.00
4	7.65	0.05	14	6.17	0.00
5	9.06	0.00	15	2.91	0.00
6	5.82	0.19	16	3.24	0.00
7	14.56	0.05	17	4.69	0.19
8	1.56	0.01	18	6.08	0.19
9	4.32	0.05	19	7.59	0.01
10	2.97	0.05	20	6.65	0.19
Total exposure			**Total PD**		
115.81			1.43		

Table 6: *CreditRisk$^+$: Portfolio overview*

Note that the total PD 1.43 is in a relation with CreditMetrics where there was the average number of defaults per scenario 1.43 (see Table 5) as well and also note that the exposure is only the LDG part of debt. The very first step is to express the exposure of every debtor j as integer number of B units. Since we set $B = 1$ we have simplified 15-bands portfolio ($M = 15$) shown in Table 7. When modeling in CreditRisk$^+$ the Table 7 is our very basic scheme that we will follow throughout the whole modeling process. Note the bands $m = 1, 10, 11, 12, 13$ and 14 are of zero values since there are not debtors

Band m	μ_m	ε_m	obligors
1	0.000	0.000	0
2	0.014	0.029	1
3	0.052	0.157	3
4	0.052	0.208	2
5	0.441	2.207	5
6	0.376	2.253	3
7	0.188	1.313	1
8	0.064	0.512	2
9	0.190	1.709	2
10	0.000	0.000	0
11	0.000	0.000	0
12	0.000	0.000	0
13	0.000	0.000	0
14	0.000	0.000	0
15	0.050	0.745	1

Table 7: *CreditRisk$^+$: Portfolio banding*

having an exposure that would classify them into these bands. But we anyway *have to* take these empty bands into account, otherwise the modeled probabilities would be unrealistic, precisely in the following way

$$\sum_{n=1}^{\infty} \Pr(\text{loss} = n) > 1. \tag{8.1}$$

Note that such a problem is given by the loss-generating function. The other point is that the bands must be however sorted out chronologically by exposure. Indeed the Table 7 is the only one possible arrangement.

Having discussed and set the underlying data we can begin in the next chapter with plain simple portfolio analysis.

8.2 Simple portfolio analysis

As mentioned before, environment of CreditRisk$^+$ allows performing more types of estimations with respect to sectorization or just on a simple portfolio with no sectorization. In this part we firstly apply the version with *no sectorization* impact. It therefore follows that in this case the portfolio is treated as homogenous and the default rates rigid, i.e. no

standard deviation of PD is taken into account. Assume the following case when we follow the given formulas for loss-generating functions, also named the recurrence relations. The probability generating function for n units of loss B has the following form (Melchiori, 2004)

$$U(n) = \sum_{m:v_m}^{M} \frac{\varepsilon_m}{n} U(n-m), \tag{8.2}$$

and for zero-loss as the starting point the following formula. Note that it is the same formula as for zero defaults

$$U(0) = e^{\sum_{m=1}^{M} -\mu_m}. \tag{8.3}$$

$U(0)$ in our case gives $U(0) = e^{-1.43} = 0.244$ and it holds that $U(0) = Q(0)$ Hence the probability of no defaults and full value of portfolio together with accrued coupons[37] (i.e. \$178.45 mm) is 24.4%. If we skipped the band $m = 1$ with zero values, then the problem would be that when $n > 0$, the probability would be infinitesimally small, but non-zero which in fact should be zero. Therefore just after start we would encounter a disproportion which piles up and based on these results the probabilities would not be very realistic and would have to be rescaled. We would have to treat $U(n)$ for $n > 0$ rather as a number than a probability.

Such problems fortunately does not encounter the probability-generating function for default events since they are not modeled via bands but only for every debtor as such. Note such a distribution is discrete. It means we only can have $1, 2, \ldots, N$ number of defaults but there are no "gaps" is such a distribution. We can observe a highpoint at $n = 1$. Precise results are given in the Table 8. Note that the table depicts the probabilities of *number* of defaults but the information about obligors' defaults are latent. The distribution of defaults is depicted in Figure 7. Knowing the distribution of defaults, we can also demonstrate the attributed losses by the Figure 8. Figure 8 demonstrates typical *Credit VaR* as the main output of CreditRisk$^+$. Bear in mind that such a chart shows the distribution of *losses* not the portfolio value as in the case of CreditMetrics. Such a distribution can be, however, remapped to a distribution of portfolio values but the

[37] Note that we can exceptionally be certain about the coupons in this case. Since if there is definitely no default, the debtors have successfully paid all their claims and hence the accrued coupons payments are of a full value.

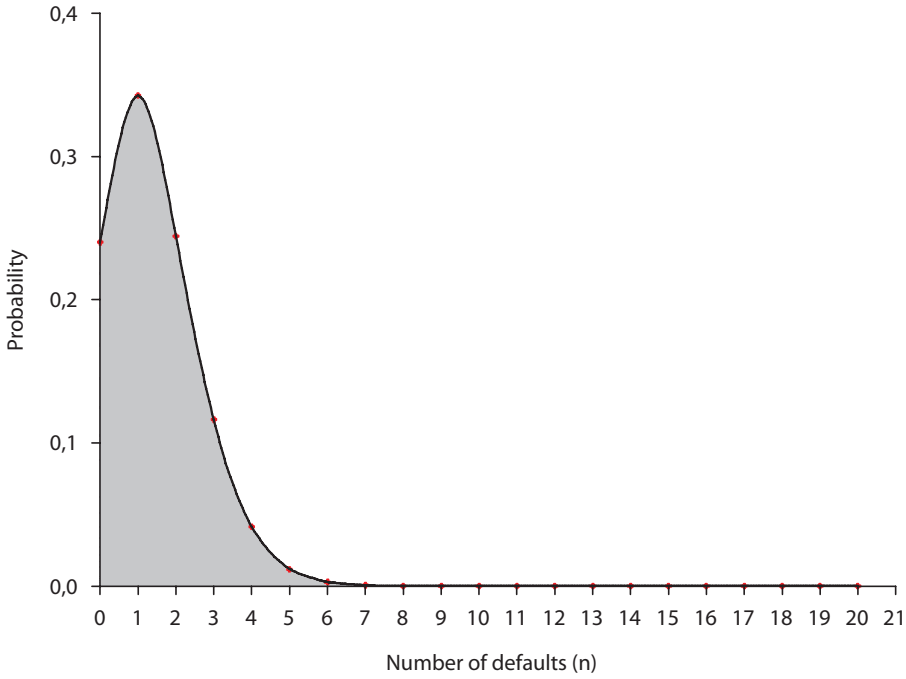

Figure 7: *CreditRisk$^+$: Distribution of defaults*

initial CreditRisk$^+$ output is a credit VaR in a loss-form. As we can see, the distribution has a typical severe "fat tail" which captures the features of credit losses. Because our portfolio is relatively small (from the amount-of-debtors point of view), the distribution has thus some peculiar properties. For instance there is a slump just after "zero defaults" point since there is relatively high probability of no default (24.4%) but just after that a loss of $1 \times B$ is unattainable as there is no such a debtor having this $1 \times B$ exposure, hence $\Pr[n = 1]$=0. And it starts to rise again when we get to the "loss-attainable" area. If the portfolio were of more debtors and/or more variable the distribution would be more smoothed. The **main problem** of comparative analysis with CreditMetrics are the **coupons**. In CreditMetrics it was possible to track-back which obligor had defaulted and utilizing this information we could simulate the coupon payments. This approach is in the case of CreditRisk$^+$ unapplicable, hence we must suffice with the assumption of lower coupon payments in worse scenarios. But because this is mathematically hardly justifiable we relax the coupons and provide results given by the original environment of CreditRisk$^+$. The results are shown in Table 9. We intentionally observe the 99.0% and 99.9% quantiles because it will be shown in later chapters how the standard deviation of *PD* changes the extreme values.

n defaults	probability	n defaults	probability
0	0.24	–	–
1	0.34	11	0.00
2	0.24	12	0.00
3	0.12	13	0.00
4	0.04	14	0.00
5	0.01	15	0.00
6	0.00	16	0.00
7	0.00	17	0.00
8	0.00	18	0.00
9	0.00	19	0.00
10	0.00	20	0.00

Table 8: *CreditRisk$^+$ plain model: distribution of defaults*

	\$ (mm)
mean loss	9.133
99% quantile	20.000
99.9% quantile	23.000

Table 9: *CreditRisk$^+$: Simple portfolio analysis*

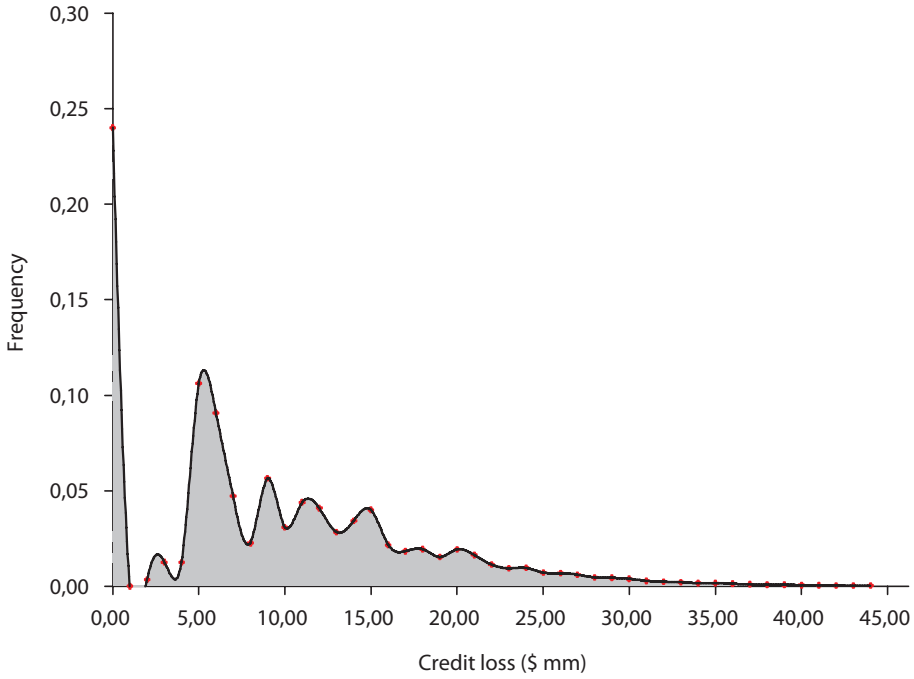

Figure 8: *CreditRisk$^+$: Distribution of losses (simple analysis)*

8.3 Extended single-sector model

In the previous chapter, we assumed no standard deviation of debtors' defaults. Such an assumption is unfortunately far from the reality. Therefore in this chapter we also incorporate the role of standard deviation of default rates. It is however very difficult to determine volatility of default so that we need some more data. We hence follow the assumption that the higher *PD* the higher the volatility. Therefore we calculate the volatility of *PD* for every debtor as the deviation of average *PD* throughout the whole portfolio (since there is only one sector assumed). Then we apply the same approach as in the previous section and employ the **Gamma distribution** for loss modeling. Note that in the preceding chapter the distribution was of *Poisson* type but since it is unable to incorporate volatilities, we must employ a different distribution, namely the Gamma distribution.

In order to calculate the loss probabilities we must incorporate the Gamma distribution as described in the theoretical section to determine firstly the defaults. After that we can basically apply two methods how to calculate the $\Pr[\text{loss} = n]$.

Firstly we may follow the classical method proposed by CreditRisk$^+$ involving solving polynomials (CreditSuisse, 1997), secondly we may use so called *Panjer recursive algorithm*

(Melchiori, 2004). We will outline both of the methods.

Firstly the polynomial approach. This approach includes calculating coefficients $a_0, a_1, a_2, \ldots, a_{M-1}$ and coefficients $b_0, b_1, b_2, \ldots, b_M$ from the formulas (8.4) and (8.6)

$$A(z) = a_0 + a_1 z + a_2 z + \cdots + a_r z^r, \tag{8.4}$$

which equals in it's aggregated form

$$A(z) = \frac{p\alpha}{\mu} \sum_{m=1}^{M} \varepsilon_m z^{\nu_m - 1}, \tag{8.5}$$

and

$$B(z) = b_0 + b_1 z + b_2 z + \cdots + b_s z^s, \tag{8.6}$$

where analogically (8.4)

$$B(z) = 1 - \frac{p}{\mu} \sum_{m=1}^{M} \mu_m z^{\nu_m}, \tag{8.7}$$

and where furthermore

$$p = \frac{\beta}{1+\beta}. \tag{8.8}$$

Since we know all the variables, we can compute the polynomial coefficients. Then we can eventually compute the loss by its generating function.

For $n = 0$ it holds that

$$A_0 = \left(\frac{1-p}{1-pn}\right)^{\alpha}\Bigg|_{n=0} = \left(\frac{1-p}{1-p\times 0}\right)^{\alpha}, \tag{8.9}$$

and for $A_1, A_2, \ldots, A_n$

$$A_{n+1} = \frac{1}{b_0(n+1)} \left(\sum_{i=0}^{\min(r,n)} a_i A_{n-1} - \sum_{j=0}^{\min(s-1,n-1)} b_{j+1}(n-j) A_{n-1} \right). \tag{8.10}$$

In this way we then compute $\Pr[\text{loss} = n]$ as $\Pr[\text{loss} = n] = A(n)$ and thus we obtain the complete distribution. For the *Panjer algorithm* we firstly define so-called *severity vector* $\boldsymbol{f_x} = (f_x(1), f_x(2), \ldots, f_x(n))$ where $f_x(n) = \frac{\mu_m}{\mu}\Big|_{m=n}$ for $n > 0$. Further we need a and b coefficients given as

$$a = \frac{\beta}{1+\beta}, b = \frac{(\alpha-1)\beta}{1+\beta}. \tag{8.11}$$

Finally, the loss-generating function according to the Panjer algorithm has a following form (Melchiori, 2004)

$$f_s(n) = \sum_{y=1}^{n} \left(a + \frac{by}{x}\right) f_x(y) f_s(n-y). \tag{8.12}$$

In our analysis we have applied the first approach to get the loss distribution. Since the total average volatility of debtors' defaults is 0.1375 the result does not differ that much from the one computed in the last section but there are, however, some particularities. Figure 9 depicts the generated distribution.

The complete results are provided in Table 10. We can see that even though the mean

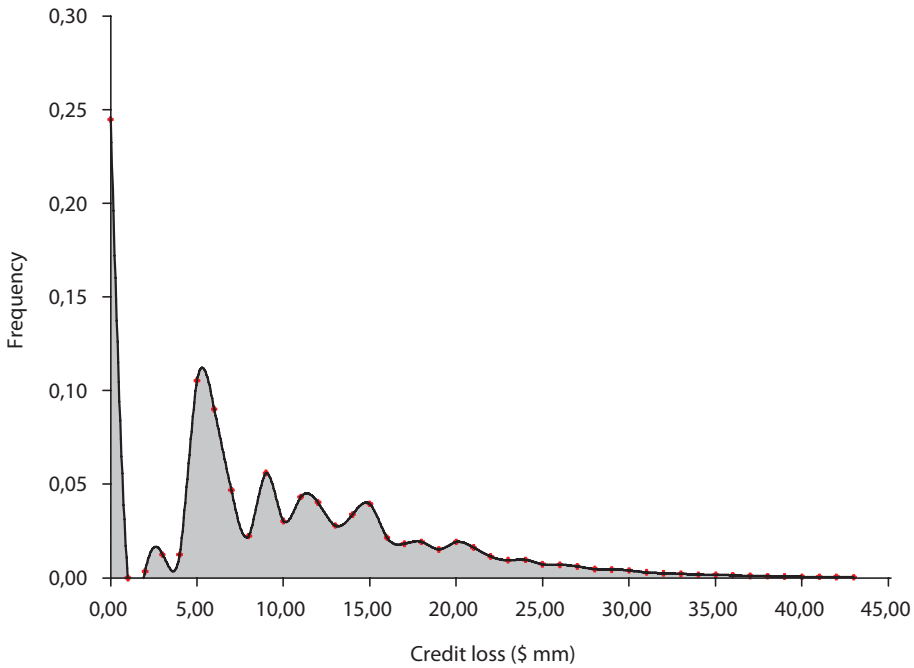

Figure 9: *CreditRisk$^+$: Distribution of losses (extended model)*

	$ (mm)
mean loss	9.133
99% quantile	34.000
99.9% quantile	45.000

Table 10: *CreditRisk$^+$: Advanced portfolio analysis*

value (and complementary mean loss) remained constant, the volatility-caused dispersion implied changes in the tails of the distribution. The quantile values differ substantially from the previous results.

8.4 Complete model with sector analysis

The sector analysis in CreditRisk$^+$ substitutes the correlation structure applied in the other models. Debtors within a sector are treated as independent subjects but there is a dependence between the sectors. CreditRisk$^+$ argues, the volatility in default rates is interconnected with the sector where the obligor operates. The main drawback is very unclear or rather uncertain way how to determine the weights of the sectors. Because of that we only can rely on an expert assessment but there is no real and reliable underlying data evidence. Since we have a portfolio of large corporates quoted on the same exchange we will apply an approach that utilizes market sensitivity. The rationale behind such an approach is that it would be impossible to define sectors and wights since the structure of our portfolio is very differentiated. Hence we rely on a common indicator but which reflects a sensitivity and also riskiness with respect to market benchmark. Therefore we decided to create "artificial" sectors created from market *Beta*. We created the sector as demonstrated in Table 11.

#	Market	Weights (%)			#	Market	Weights (%)		
	Beta	**Own**	**Sector 1**	**Sector 2**		**Beta**	**Own**	**Sector 1**	**Sector 2**
1	1.51	50	50	0	11	2.37	25	25	50
2	1.32	50	50	0	12	0.66	100	0	0
3	1.27	50	50	0	13	1.47	50	50	0
4	1.12	50	50	0	14	1.39	50	50	0
5	1.69	50	50	0	15	0.59	100	0	0
6	1.40	50	50	0	16	1.48	50	50	0
7	2.77	25	25	50	17	1.88	50	50	0
8	1.70	50	50	0	18	2.43	25	25	50
9	1.33	50	50	0	19	1.72	50	50	0
10	2.07	25	25	50	20	2.42	25	25	50

Table 11: *CreditRisk$^+$: Sectors establishment*

Let us now clarify the sectorization principle shown in the Table 11.

For $\beta \in (-\infty, 1>$ there is rather **low market sensitivity** and hence we attribute only own risk-driver with weight 100%.

For $\beta \in (1, 2>$ there is **above average market sensitivity**, therefore own risk-weight is only 50% and "market" risk, represented by artificial Sector 1 is also 50%.

For $\beta \in (2, +\infty)$ the firm's **market sensitivity** (and riskiness) **is extreme**, therefore own risk weight is only 25%, Sector's 1 also 25% and we further introduce Sector 2 with 50% risk-weight to represent a participation in a very risky market.

Having the participation weights we define average weights in bands. Then, we define for every sector volatility and we can calculate the loss applying polynomial solving approach. We obtained the following results represented by Figure 10 and numerical data are given in Table 12.

	$ (mm)
mean loss	9.133
99% quantile	34.000
99.9% quantile	47.000

Table 12: *CreditRisk$^+$: Sector analysis*

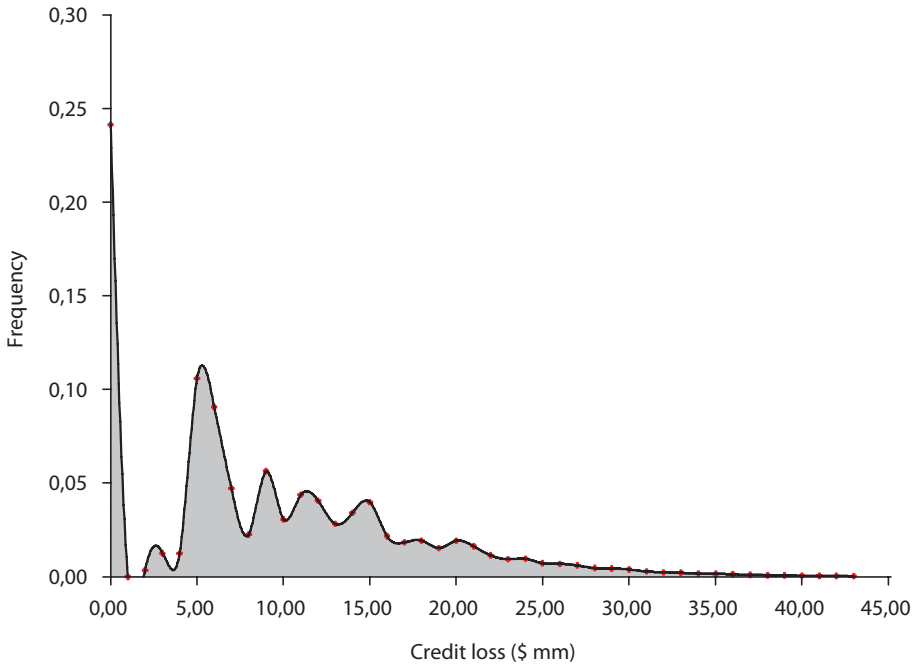

Figure 10: *CreditRisk$^+$: Distribution of losses (sector analysis)*

The only difference of this CreditRisk$^+$ highest-sensitivity analysis is the difference in 99.9% quantile slight shift from \$45.000 mm in single sector model to value \$47.000 mm in multi-sector model. It just underscores that the correlations (via sector analysis) are not causing any significant changes in the Credit VaR. Compare for instance with the CreditMetrics with adjusting the overall correlation impact.

Generally one can hardly rely on the results even though they seem to be relatively consistent since they are created on very unfounded assumptions (e.g. the *Beta* approach to the sector analysis) and hence CreditRisk$^+$ should not be used for correlated portfolios.

8.5 CreditRisk$^+$: The bottom line

In the last several chapters we have analyzed CreditRisk$^+$ portfolio model. Generally, we can not definitely say the model is not suitable for risk evaluation but certainly it is not suitable for company debt evaluation. There are many reasons giving the evidence such a claim. *Firstly*, there it is necessary to round the exposures so that we can create integer-sized exposure bands. We reckon that in the era of modern computer technology it is unnecessary wasting of accuracy. *Secondly*, the model approximates the losses, e.g. there is always very small possibility of incurring loss (subtle or large loss) that in fact is

not attainable and also it allows multiple defaults of one debtor that can hardly occur. As a *third* remark, the correlations are not solved properly. The substituting **sector analysis is strictly human judgement**-dependent and hence such results might be very inaccurate. Another drawback is **no coupon payments** incorporation. We would have to use a very primitive and insensitive own framework to incorporate the coupons to enable us to compare results with the other models. On the other hand, the requirements for inputs we had to fill were very undemanding. As a main positive we deem to be the absence of simulation (from the computational point of view, not the one related to results) but from other point of view it makes the model very rigid and inflexible. Unfortunately our portfolio consists of corporate debt therefore we can not assess the predictive power for a numerous retail portfolio, where the model should be more efficient.

Taking into account all the listed features and melting them together, we do not recommend CreditRisk$^+$ for evaluation of sparse corporate debt portfolios. However, it might be used as a supporting decisive tool.

9 Applied KMV model

Our very last interest is in KMV model. Modeling over the KMV-model methodology differs from the both preceding models in the sense it does not use classical rating data but relies clearly on capital market. Hence the model is very sensitive but also easily offended by data inconsistency or data that are not perfectly accurate[38]. We therefore put great emphasis on **model calibration** as it is the crucial aspect of the modeling. Note that in case of CreditMetrics the model structure was more or less deterministic given whereas KMV demands a serious attention in this issue. The admirable feature of KMV is definitely also the fact that it essentially creates *self-produced* default probabilities contrary CreditMetrics and CreditRisk$^+$ where the *PD*s were essential inputs and we basically only worked out their *combination* resulting in modeled defaults and losses. The last point to be mentioned on the KMV's account is the extreme precision to even a slight change in debtor's fitness. Even in CreditMetrics[39] we can observe *discrete behavior* such as the "slump(s)" in portfolio value (see Figure 5 for commemoration). KMV exercising its market-sensitivity approach does not encounter such discrete features since it can perfectly evaluate no matter how small change in debtor's fitness[40].

Knowing all the necessary features we will now go through the whole KMV's modeling process that comprises the following steps: data mining & detailing, *DD* determination, *EDF* derivation, translation of *EDF* to risk-neutral probabilities and eventually the portfolio simulation. We will also introduce several extensions to upgrade the model abilities/accuracy similarly to CreditMetrics as described in the Section 7.4.

9.1 Data collecting

Since the KMV model is capital market data dependent, we firstly need to discover debtors' capital structure characteristics. Those data are gathered over *Yahoo! Finance* and *Google Finance*, and in some cases also *CNN Money*. The capital structure of debtors is up to

[38] naturally all the models are sensitive to the data validity but in case of KMV this aspect is even more stressed since KMV uses single indicators such as *market growth* which subsequently determines the whole portfolio analysis to a large degree

[39] which is said to be marked-to-market

[40] note that even though plain KMV is *default-mode* model its valuation extension is perfectly marked-to-market

the date 31 Dec 2009[41]. From the capital market we to the addition obtained market growth[42] $\mu_M = 11.4\%$, market β_j for all detors $j = 1, 2, 3, \ldots, J$ and the time structure of risk free rate r we already have mined for CreditMetrics[43].

After we collected the *essential* capital market data, we applied formulas (4.4) and (4.8) and iteratively derived A_0 and σ_A. Note that both the A_0 and σ_A are relative to DPT debt level therefore these values can not be deemed to be the "real" but estimated values. Used time period for estimation was similarly CreditMetrics $T = 1$ and the risk free rate $r = 0.3\%$. Finally the collected data for KMV are presented in Table 13.

[41] to guarantee that the capital structure is up to the date 31 Dec 2009 we in few cases had to browse their company web pages and checked their audited annual report

[42] since the debtors are quoted on NYSE we use $\mu_M = \mu_{(NYSE)}$ which is represented by *NYSE composite index*

[43] note that we only have separate time points, e.g. time structure for integer-years and hence we will have to derive continuous term structure

#	E_0 ($ mm)	σ_E	A_0 ($ mm)	σ_A	DPT ($ mm)	Cur L ($ mm)	Lt L ($ mm)	β	μ
1	2009	0.539	6539	0.166	4545	1783	5524	1.51	0.165
2	817	0.662	6713	0.081	5914	2302	7224	1.32	0.144
3	12400	1.388	34652	0.498	22466	6281	32370	1.27	0.139
4	7316	0.308	15503	0.145	8212	2007	12410	1.12	0.123
5	1441	0.454	2923	0.224	1487	815	1343	1.69	0.184
6	436	0.484	6119	0.034	5700	1702	7996	1.40	0.153
7	605	0.869	1380	0.383	783	357	852	2.77	0.299
8	3391	0.506	11628	0.148	8264	6877	2773	1.70	0.185
9	986	0.396	1440	0.271	455	103	704	1.33	0.145
10	1191	0.524	1973	0.316	785	370	829	2.07	0.224
11	335	0.719	1432	0.168	1101	333	1537	2.37	0.257
12	9998	0.165	25166	0.065	15214	12800	4827	0.66	0.074
13	3641	0.333	6617	0.183	2985	2456	1057	1.47	0.160
14	6221	0.365	58725	0.039	52662	45137	15050	1.39	0.152
15	23810	0.179	34399	0.124	10621	9281	2680	0.59	0.066
16	9690	0.453	15280	0.287	5607	3022	5169	1.48	0.161
17	1155	0.797	6450	0.143	5311	2896	4830	1.88	0.204
18	501	1.414	1992	0.355	1496	446	2101	2.43	0.263
19	2807	0.569	11070	0.144	8290	4389	7802	1.72	0.187
20	279	0.780	1712	0.127	1438	502	1873	2.42	0.262

Table 13: *KMV: Capital market data collection*

What we need to do firstly is to evaluate the portfolio at $t = 0$ to enable us to compare the initial value with the value at desired time horizon $T = 1$. However for the starting point $t = 0$ valuation we also need complete (continuous) structure of risk free rate r_t, where $t > 0$. Since KMV is very sensitive to data inputs we can not use the risk free rate approximation approach exercised in the Section 7.2 for the reason such an approximation is not perfect. Hence in KMV we need to use a different approach. To achieve such a structure we decided to employ curve fitting by *interpolation using Lagrange polynomials.* Interpolated risk free rate curve is depicted in Figure 11. See the curve fully captures the term structure and hence we can take advantage of its continuous character (Ramaswamy, 2003). Also note the curve is defined till $t = 7$ to cover the bonds maturities throughout the whole portfolio. (See Table 1 for the portfolio qualities overview). We now have full data to evaluate the portfolio at $t = 0$ and that will be the subject matter in the next

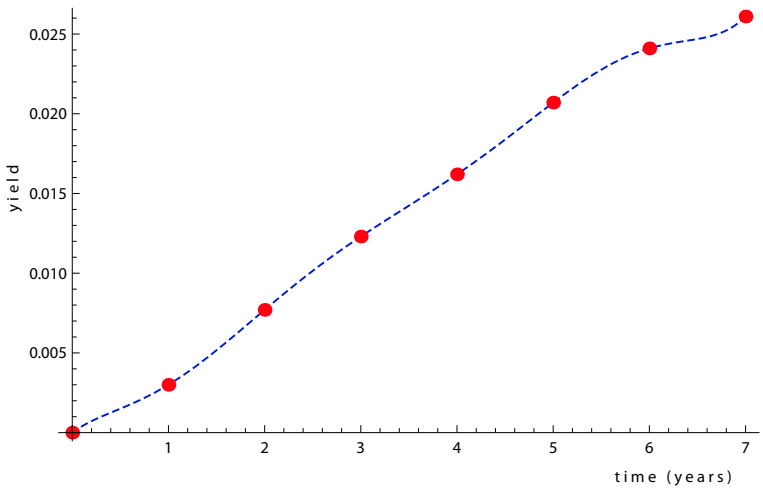

Figure 11: *Interpolated risk free rate curve*

chapter. To the addition we will also need to get some more additional data to complete the analysis but it will be discussed closer in intended chapter.

9.2 Present value determination

The portfolio present value determination at $t = 0$ is not that straightforward as in the case of CreditMetrics therefore we dedicate one entire section to this problem. The process of PV_0 estimation resides in separating future cash flows of bonds into risk free and risky part that are finally totaled. See the formula (4.15) that describes it precisely. The calculation of risk free PV is straightforward. On the other hand calculation of the risky part of cash flow is more demanding.

What we have to do for calculation of the risky PV part is to determine DDs, translate them to EDFs and subsequently determine risk neutral probabilities $PD_{rn}(t_i)$ which are eventually used for discounting. Note that the DD (and hence EDF and also $PD_{rn}(t_i)$) structure is deterministic when we evaluate at $t = 0$ hence we similarly CreditMetrics assume the debtors will not default but instead of default their risk neutral discounting rates $PD_{rn}(t_i)$ incorporate the rising PD. In fact KMV methodology is so astute that the "almost certainly" defaulting debtors are "penalized" by very high values of $PD_{rn}(t_i)$ in distant discounting periods.

Putting the theory into effect we calculate DD for every debtors' discounting period[44] and then we have to map for every DD the EDF as described in the Section 4.3. But

[44]note that since the portfolio is real the discounting periods are different

such a mapping requires additional input, namely the $DD \rightarrow EDF$ mapping algorithm. Precisely we are looking for explicit form of formula (4.11). The problem is that complete $DD \rightarrow EDF$ database is the main know-how of KMV company and hence it is unfortunately not freely accessible to public. Therefore we must do with a simplified generalized scheme outlined in (Sironi and Rosti, 2007) (see Table 14). A different $DD \rightarrow EDF$ mechanism resulting in substantially different results will be introduced in Appendix. Once again we encounter another problem because the $DD \rightarrow EDF$ structure is discrete but we need continuous form. The discontinuity can be solved by the same interpolation mechanism

DD	0	1	2	3	4	5	6
EDF	20%	8%	3%	1%	0.40%	0.07%	0.04%

Table 14: *DD to EDF mapping*

as in the case of risk free rate. Applying Lagrange's polynomial interpolation we obtain a continuous scheme as shown in Figure 12. The derived relation $DD \rightarrow EDF$ takes then

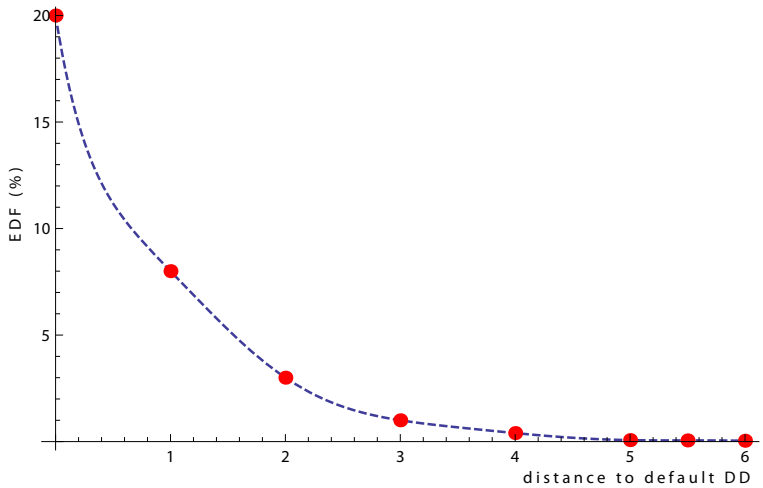

Figure 12: *Interpolated DD→EDF scheme*

the following polynomial form:

```
EDF(DD)=
=20.-33.897DD+51.823DD^2-51.006DD^3+29.484DD^4-10.439DD^5+2.340DD^6-0.334DD^7+0.029DD^8-0.001DD^9.
```

Having expressed the explicit DD to EDF relation we can step forward to $PD_{rn}(t_i)$ derivation. We evaluate every debtor in time period t_i (related to the coupon payment) and following the formula (4.12) we calculate $PD_{rn}(t_i)$ for every discounting period. Now we

mechanically calculate the risky PV_0 and total it with the risk free PV_0. The PVs (total) for all debtors we have achieved are shown in Table 15. We can see that the estimated

#	Par	Coupon	LGD	PV riskfree	PV risky	PV risk free + risky
1	10.50	7.4%	85%	**1.875**	**9.258**	**11.132**
2	5.00	6.9%	95%	**0.299**	**4.953**	**5.252**
3	7.00	6.0%	74%	**2.274**	**5.384**	**7.658**
4	8.50	8.0%	90%	**0.974**	**8.555**	**9.529**
5	14.00	5.6%	65%	**5.662**	**8.797**	**14.459**
6	6.00	8.8%	97%	**0.209**	**6.194**	**6.404**
7	20.50	6.5%	71%	**7.033**	**13.241**	**20.273**
8	3.00	6.3%	52%	**1.721**	**1.508**	**3.229**
9	6.00	1.9%	72%	**1.730**	**4.308**	**6.038**
10	5.50	9.9%	54%	**2.889**	**3.213**	**6.102**
11	11.00	6.3%	48%	**6.788**	**4.863**	**11.652**
12	7.00	7.4%	65%	**2.949**	**5.441**	**8.390**
13	6.00	6.0%	65%	**2.561**	**3.974**	**6.535**
14	9.00	5.8%	69%	**3.405**	**6.020**	**9.425**
15	4.50	5.3%	65%	**1.772**	**3.278**	**5.051**
16	5.00	5.2%	65%	**1.979**	**3.328**	**5.307**
17	7.00	7.3%	67%	**2.775**	**4.675**	**7.451**
18	8.00	5.3%	76%	**2.212**	**5.384**	**7.596**
19	13.50	6.6%	56%	**6.527**	**7.469**	**13.996**
20	7.00	7.5%	95%	**0.453**	**6.548**	**7.001**
Total	**164.00**			**56.087**	**116.390**	**172.478**

Table 15: *KMV: present values at $t = 0$ (in \$ mm)*

total PV **\$172.478 mm** in KMV is very similar to the one calculated by CreditMetrics (see Section 7.2 for more details) which was **\$173.31 mm**. Having calculated the PV at $t = 0$ we can turn our interest to the future portfolio value which covers the next chapter.

9.3 Portfolio value at $T = 1$ and simulation methodology

Our primary target is to estimate the portfolio value distribution in one year horizon (i.e. $T = 1$). The rationale behind the forward valuation of bonds in KMV is in fact just shifted pricing approach from previous chapter. The pricing methodology hence does not involve any new aspects. However, we use forward risk free rates for discounting but above that

there is no change. Yet what is important for the valuation is the starting point of asset value A since it ceteris paribus has incidence on DD (see formulas (4.4) and (4.10)). What we are going to simulate are the values $A(T)$ for $T = 1$ that will determine future value of bonds. And also the $A(T)$ value determine whether the debtor defaulted or not[45] from starting point $t = 0$ to the critical time $t = T = 1$.

Let us now discuss the simulation process. We will follow the steps mentioned in Section 2 about CreditMetrics to incorporate correlations. The correlation structure is known since we have already used it. However, instead of the thresholds applied in CreditMetrics the generated random numbers are transformed into correlated probability as an input for the lognormal property sampling (see formula (4.9)). Hence similarly CreditMetrics we firstly generate future scenario by $T = 1$ and then we divide our interests into 2 directions: backward and forward. In every simulation we firstly sort out debtors into two classes: defaulted and non-defaulted. After that we backwardly track coupon payments. We apply the very same approved approach for defaulted debtors as in CreditMetrics (see Extension 1 in Section (7.4.1)). When the debtor survives by simulated $T = 1$ (i.e. $A(T) \geq DPT$) we based on $A(T)$ evaluate his future[46] cash flows for $t > T \mid T = 1$. The simulated $A(T)$ is therefore **crucial** since it determines both default/non-default and debtor's health represented by initial position for the future which the discounting takes into account. We basically for every simulated $A(T)$ determine DDs for the outstanding discounting periods and then we discount analogically the technique described in Section 9.2. What we do now know are the precise forward risk free rates. Forward rates in continuous form are derived in two steps. Firstly we follow the formula (2.2) to get forward discrete values of risk free rate. Then we apply once again the Lagrange interpolation to derive the continuous form. The forward risk free rate curve is shown in Figure 13. We will as well as in the case of CreditMetrics simulate 50 000 scenarios and then evaluate the portfolio. The approach of data aggregation and determining the statistical indicators is the very same as for CreditMetrics. Let us now concern with obtained results.

[45] we evaluate the conditions $A(T) < DPT$ (default) and $A(T) \geq DPT$ (non-default)

[46] the forward direction

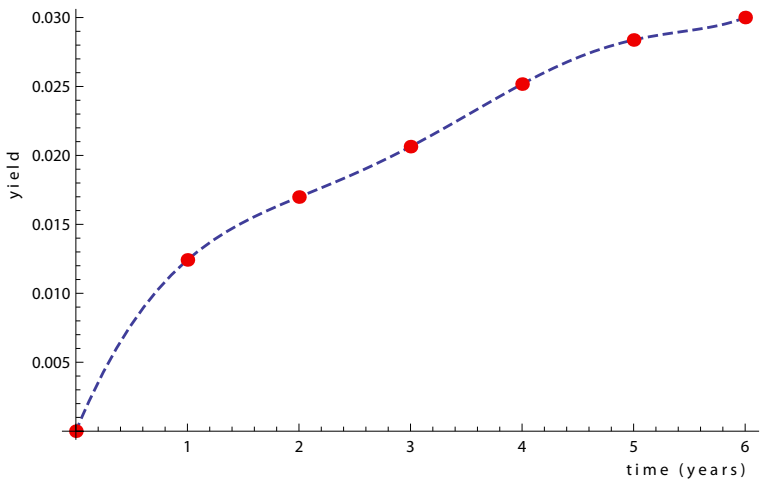

Figure 13: *Interpolated forward risk free rate curve*

9.4 KMV: Simulation output

In this chapter we are going to analyze the simulated results and also moreover we will also try to identify and solve some shortcomings we noticed during the modeling. Firstly let us comment the Table 16. The results are relative consistent with results obtained from CreditMetrics (see Table 5) but there are some particularities. Firstly see the mean number of defaults in each run which is **1.043** in KMV case contrary **1.430** of CreditMetrics whence the mean portfolio value in KMV is lower than in CreditMetrics. It can be explained by the character of KMV since it argues that the actual transition probabilities are much higher than those given by classical transition matrices but on the other hand the default probabilities of the "junk bonds" are usually lower. Hence there are less defaults but "higher migration frequency" resulting in lower mean portfolio value. The quantile values are quite similar as those of CreditMetrics. But in any case the extreme portfolio values (distribution tails) are severer in CreditMetrics. For analyzing the distribution character see Figure 14. The distribution is nicely smoothed, no granularity, slumps or anomalies are to be detected contrary CreditMetrics (see Figure 5). The distribution is very close to perfect lognormal distribution shape. It is largely owing to absolute perfect sensitivity to debtor's health subsequently reflected in bond pricing. Note that in CreditMetrics there always has to be exceeded a threshold value to trigger change in debtor's health. Therefore KMV model is suitable even for valuation of sparse portfolios.

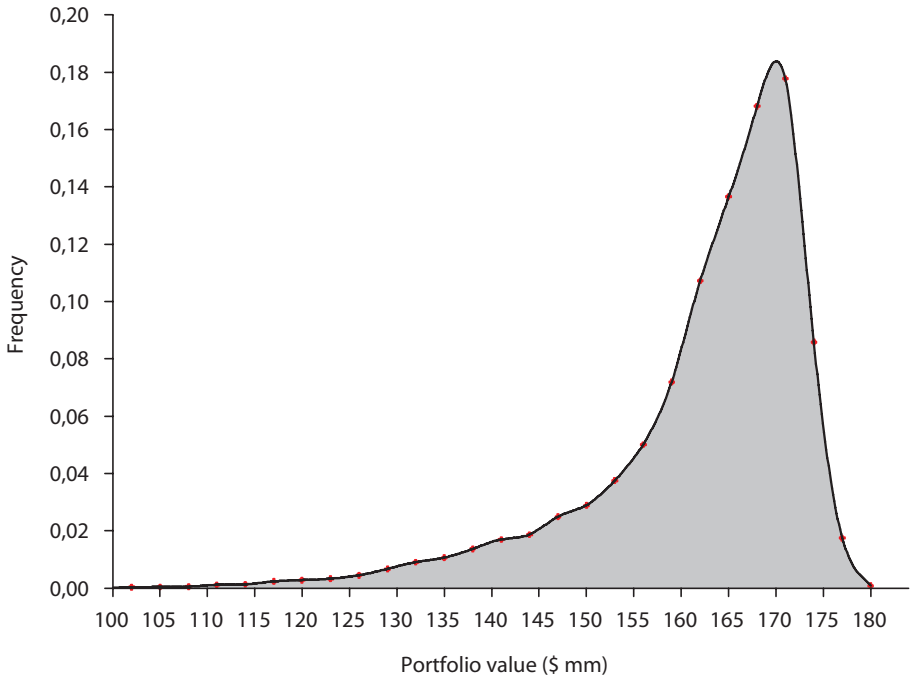

Figure 14: *KMV: Portfolio value distribution*

Even though the results seem to be certainly convincing we so far have been leaving out of consideration one very important aspect, namely the *DPT* was assumed to be constant not volatile. The plain KMV model does not concerns about such an issue since it is not that important for shorter time periods but we have claims with almost 7-year maturities so that we should be concerned. The solution is described in the next section.

Summary statistics			
Runs	**Mean value**	**Standard Deviation**	**Unexpected loss**
50 000	160.726	11.364	39.93
Mode	**Median**	**1% quantile**	**99% quantile**
167.38	163.96	120.800	174.769
Worst case scenario		**Best case scenario**	
80.80		178.77	
Average number of defaults in every run			
1.014			

Table 16: *KMV: portfolio simulation results ($ mm)*

9.5 Own KMV model extension: volatile *DPT*

Following the previously outlined problem of *DPT* we in this chapter suggest implementation of ***DPT* volatility**.

Let us firstly precisely define the issue. Assume we have a bond with very distant maturity. What KMV does when it valuates the cash-flow is it reprices every risky cash flow part with risk adjusted discounting rates. The risk-adjusted discounting rate is derived from *DD* mapped to *EDF*. The crucial subject matter is the *DD* since it reflects debtor's health. Apart from A and σ_A (and general market data) it also takes into account *DPT* value as a very important constituent. As A is revaluated throughout the simulation the *DPT* is not in the plain version which is in fact hardly close to the reality since normally the debt (represented by *DPT*) oscillates. Hence the simulation in environment of long-maturity bond should also cover this oscillation. Figure 15 graphically captures the idea of the proposed extension. Such an extension is incorporated into our

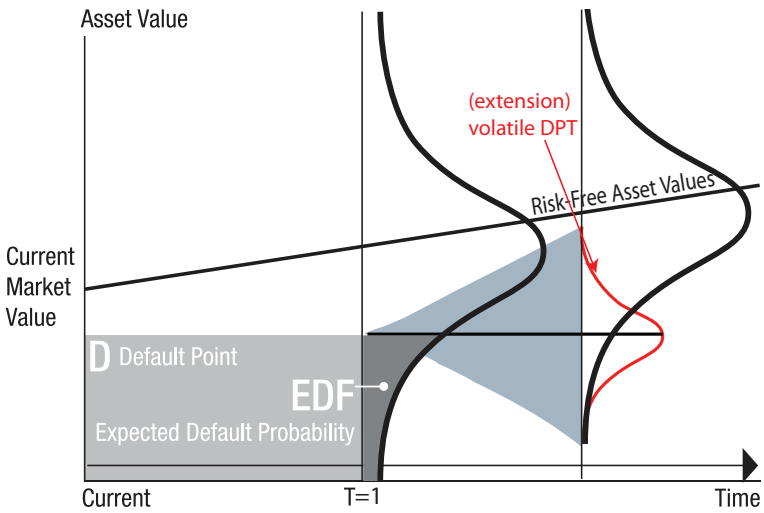

Figure 15: *KMV: DPT volatility extension. Source: (Colquitt, 2007)*

model but in the analyzed simulation it was inactive to firstly demonstrate the model in its plain version. The modeling rationale behind the *DPT* oscillation is defined very simply as $DPT_t \sim N(DPT_{t-0.5}, \sigma^2_{DPT})$. As it can be seen our model always loads the preceding *DPT* level[47] and assigns new one which is subsequently evaluated by the *DD*-determination process. The only problem might be the volatility of debt that might differ

[47] note that the DPT revaluation takes place for every 0.5 year since the debtors pay coupon semiannually so that the revaluation is more frequent and hence can simulate the changes in a very flexible manner

company to company but for simplification we use a volatility proportionate to the market growth.

Above that we also use the same extensions as listed in the Section 7.4. To give a real evidence to the KMV's *DPT* extra extension it is the subject matter of the following chapter.

9.6 KMV: The full model

In this part we just briefly stress out the main differences between the previous model with rigid *DPT* level and the extended one with ***DPT* fluctuations incorporated**. Graphical

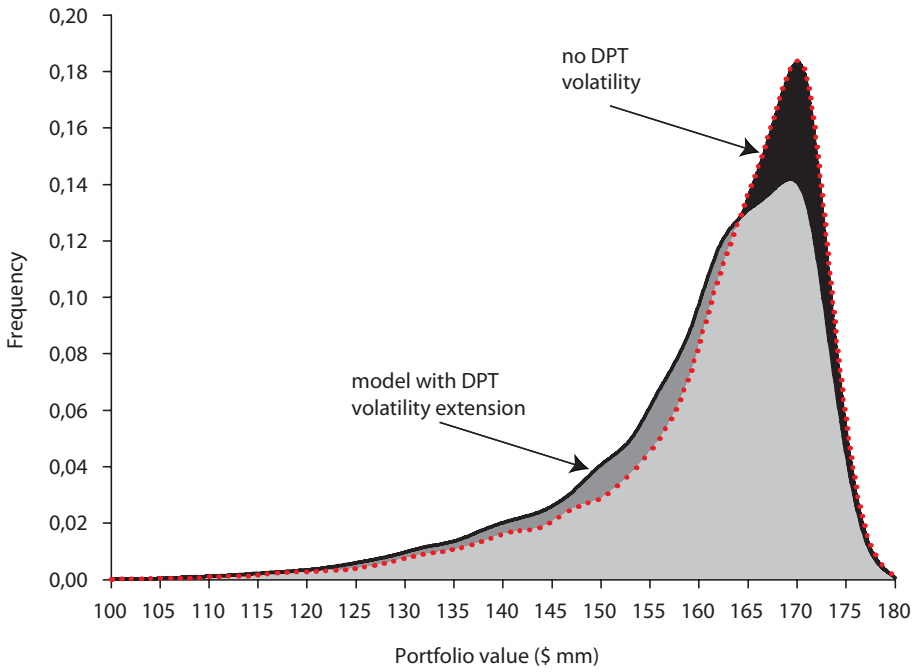

Figure 16: *KMV: DPT and no-DPT KMV model*

comparison of both model depicts Figure 16. What is discernible at the first sight is the "less stabile" shape of the *DPT*-adjusted model. Even though the most results still peak around the area of $160 mm to $175 mm it is remarkable that also significant portion of distribution values has shifted to lower values. Analyzing the table of numerical results the mean portfolio value is slightly lower than the previous plain model also with higher standard deviation. What is remarkable is the growth of mean value of defaults that has

Summary statistics			
Runs	**Mean value**	**Standard Deviation**	**Unexpected loss**
50 000	158.598	12.091	39.897
Mode	**Median**	**1% quantile**	**99% quantile**
168.051	161.680	118.771	174.506
Worst case scenario		**Best case scenario**	
87.656		178.609	
Average number of defaults in every run			
1.392			

Table 17: *KMV: simulation results (DPT-volatility module on)*
($ mm)

risen from 1.014 to 1.392[48]. Hence in some cases the raised DPT level triggered defaults that in the previous version would not occur. On the other hand for the other debtors the "default position" might have improve so that they are eventually more distant from default. The result is hence the discovery that even though there were recorded more defaults the final mean portfolio value was not that much affected since for the other debtors the situation improved. It also results into less severer tail (see the worst case scenario with volatile DPT is valued $87.656 mm value whence in the plain model it reaches much worse $80.800 mm value.

[48] see the mean amount of defaults in improved KMV is very close to the number we got in CreditMetrics (which was 1.43 defaults per scenario)

9.7 KMV summary

KMV has proved it is a well-designed model with a high focus on market dynamics. The important general difference is it uses rather a *point in time* point of view contrary CreditMetrics that uses rating and results more likely in a *through the cycle* methodology. It can be proved simply by the claim that due to problems of financial markets in the last time and significant growth the KMV attributes **higher uncertainty**[49] to debtors that is immediately reflected in the **risk-adjusted discounting**. Even though we can not explicitly say KMV is better than CreditMetrics it definitely provides very sensitive analysis. Comparison with CreditRisk$^+$ is difficult since both of the models are totally different and what is more the impossibility to incorporate coupon payments to CreditRisk$^+$ makes it even more complicated. We therefore can hardly compare numerical results. On the other hand we can definitely say those models are each on a different pole. CreditRisk$^+$ benefits from lack of information and numerosity whence KMV benefits from accurate information, is able to evaluate a single debtor and can not be used in any environment not having such a quality. The main KMV's drawback is simply the fact it can be only applied to corporates quoted on capital market.

[49] if the market started to grow it would however imply lower discounting rates than 'slower' CreditMetrics

10 Explicit model-to-model comparison

In this last chapter we are going to compare all of the analyzed models and provide clear model-to-model results. All of the models will be demonstrated in their full version forms, to capture their power in the most developed form. To be more concrete, the CreditMetrics model is tested with all possible extensions, CreditRisk$^+$ via full sector analysis and KMV similarly Section 9.6 with *DPT*-volatility module activated. What remains unsolved are the coupon payments in CreditRisk$^+$ so that they are relaxed and the reader must take such a fact into account. Note that the *paid* - *unpaid* coupon payments difference makes approximately \$10.6 mm. Moreover we also apply more sensitive filter so that the final graphical results are more distinct. To make the analysis complete a translation of CreditRisk$^+$ Credit VaR output into the portfolio value form has to be done to provide results comparable with the other models.

Figure 17 finally shows the ultimate overview comprising all tested models. Numerical data presented in Table 18 are those used in preceding chapters so that there are no new results. The noteworthy discovery in Figure 17 is definitely the relative analogy of Cred-

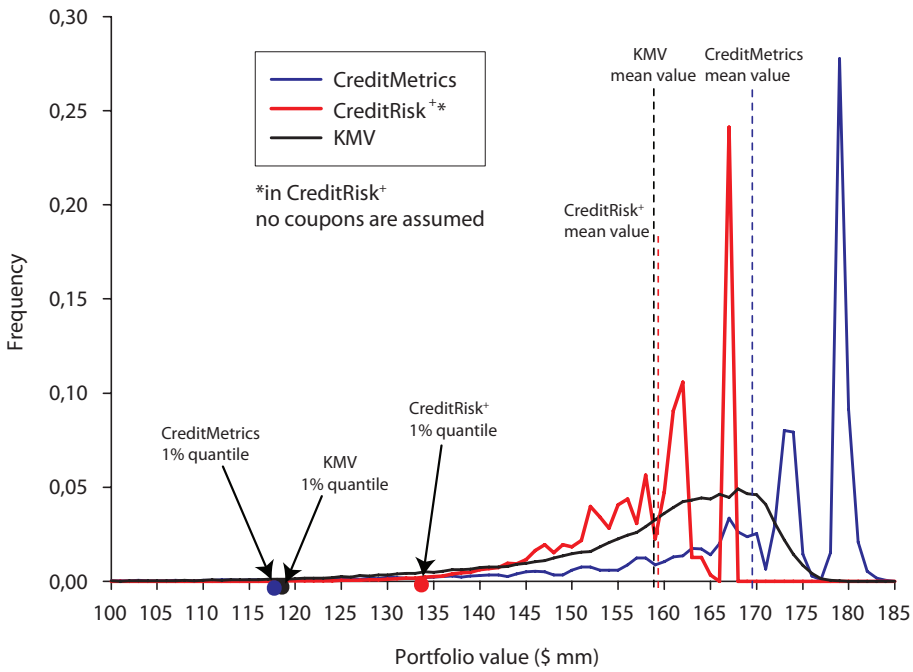

Figure 17: *Ultimate model comparison*

itMetrics and CreditRisk$^+$ portfolio value distribution. Note that in fact the similarity

ought to be even higher since CreditRisk$^+$'s values are shifted to the left since we could not incorporate coupon payments. We reckon that the clear similitude of both plots is simply caused by the very same rating agency inputs. On the other hand the plot of CreditRisk$^+$ is actually of a discrete form but for comparison shown as continuous and this fact could also influence the character. In any case both CreditMetrics and CreditRisk$^+$ are apparently less sensitive to the risk than KMV whose portfolio value distribution plot shape is very smooth and resembling very closely the lognormal curve shape. The numerical results

	Mean value	**1% quantile**	**99% quantile**	**Unexpected loss**
CreditMetrics	169.141	118.069	180.755	45.931
CreditRisk$^+$	158.867	134.000	168.000	24.867
KMV	158.598	118.771	174.506	39.827

Table 18: *Ultimate model comparison ($ mm)*

have already been compared in chapter related to each depicted model but on a general level in terms of portfolio value the most critical model was KMV largely owing to PIT point of view which incorporates current market uncertainty and therefore "penalizes" worse debtors' qualities more than the other models. The severity of 1% quantile values is for CreditMetrics and KMV almost the same contrary CreditRisk$^+$ whose 1% quantile is quite moderate. We attribute this quality particularly to relaxed approach to correlation structure. Note that the worst (and oppositely also the best) case scenarios are induced by highly correlation-affected scenarios.

Having compared the portfolio credit risk models directly we can finally summarize our observations in the conclusion.

11 Conclusion

The main objective of this thesis was a rigorous general analysis of commonly used portfolio credit risk models CreditMetrics, CreditRisk$^+$ and KMV. We have provided an insight not only on theoretical but on practical level as well how these models operate in current highly turbulent debt market environment. What is more, our implicit task was also to verify many assumptions that are related to portfolio credit risk models.

Firstly our main interest resided in a deep scrutiny of the theory related to the screened models. The operating mechanisms were clarified and apart from that we have listed the main positives and however also several drawbacks the portfolio models are characterized by. One could even during the theoretical explanation given in the theoretical part discover what subtleties and rationales are concealed under the models' construction and based on that could create own opinion about every single model. Having described the models' mechanics we demonstrated their power on a portfolio.

In the applied practical part we have defined to work out a real demonstration of selected models. The very first issue was the portfolio sample assembly which we created from real-world corporate bonds to exhibit models to real-world environment. Then we performed modeling in environment of each single model. It was in some cases inevitable to accept certain level of compromise otherwise we would not be able to make any advance. Naturally, accepting a compromise is always a trade off for exact results therefore we strived to minimize such steps.
In spite of the compromises the generated results verified the theory (and implied assumptions) very faithfully. Putting all the theory and practice together and commenting the obtained results we may claim that KMV model is a cutting-edge model in debt evaluation but on the other hand needs perfect capital market data.

The greatest positive is simply the fact that KMV contrary the models can evaluate even a slight change in debtor's fitness whereas CreditMetrics and CreditRisk$^+$ rely defined on rating-based data so that these models are not that perfectly sensitive.

CreditMetrics is a very robust model that certainly performs very well when the rating assigned by a rating agency is set adequately and reliably reflects debtor's health. The main positive when compared to KMV is no necessity of capital market data but it is logically less reactive.

The remaining analyzed model was CreditRisk$^+$ which is rather suitable for evaluation

of different type of debt (retail) but we also put a lot of effort to employ this model in large-corporates sparse debt portfolio[50]. The results are not that credible as those given by KMV or CreditMetrics but as we have shown in the last chapter the results were not that distant from CreditMetrics. Anyway, we do not recommend to use CreditRisk$^+$ for such a portfolio.

The final verdict about suitability is that we would *recommend* to use **KMV** if **capital market data** are at disposal, **otherwise** we would use **CreditMetrics** and if the portfolio is **numerous and homogeneous** we would also suggest using employ **CreditRisk$^+$** whence KMV could hardly be used.

Nevertheless, occurrences of the last two years have proved that even well-designed models can not predict losses perfectly or at least not enough in advance. If there is a market downturn debtors become more correlated and hence the severity unexpectedly rises, causing severer losses when there are correlated defaults. Financial institutions should never rely on predicting power of those models absolutely and to the addition a security cushion should be created to cover unexpected events. To the contrary the history has proved that the important role of well-calibrated quantitative credit risk evaluation models is certainly justified and so far it is the best possible way how to manage credit risk.

[50] portfolio characterized by high nominal values but only of a few issuers

Appendix: Alternative approach to DD to EDF mapping

This Appendix is an extra addendum to the Section 9.2. Since the *DD→EDF* is a black box[51] true explicit relation for the *DD→EDF* conversion will be in terms of approximation and data fitting. It needs to be noted that the derived *EDF* is crucial since it uniquely determines future value of the debt hence we find it necessary to fully specify a way how we derive it. Since in the Section 9.2 we used data from (Sironi and Rosti, 2007) (see Table 14) but there are also a different data at disposal we want to provide the reader also with an alternative approach and show how radically the results based on two different *DD→EDF* schemes differ.

In this Appendix we are using data sample of 100 companies (source: (Jezbera, 2010)) for which the *DD→EDF* was defined. What we need for portfolio simulation is either an explicit relation *EDF(DD)* or a numerous set of conditions such as

$$\forall DD \in < x, x + \Delta >: EDF(DD) = y. \tag{11.1}$$

We will combine both approaches otherwise the results would not be accurate. Since there is an inverse *DD* vs. *EDF* relation we must firstly search for a kind of interpolating inverse function that will be subsequently put under conditions but we can not use the same polynomial approach as in Section 9.2 because the character of these data is different and the inverse-monotony character od *DD→EDF* would be thus violated. We therefore have 2 main requirements for the interpolating curve quality: *monotonic* and with *R-sq* statistic close to 1. After several observations we have discovered that *Gaussian interpolation* of k-th level has these qualities. More precisely we were searching for a *Gaussian function* $f(x)$ such that

$$f(x) = \sum_{i=1}^{k} a_i e^{-\frac{(-b_i+x)^2}{c_i^2}}. \tag{11.2}$$

where a_i, b_i, c_i are parameters for $i = 1, 2, \ldots, k$ and $x = DD$. Iterative suggestion proposed searching for coefficients with $k = 7$. After follow-up iterative search putting $k = 7$ we obtain the following results:

[51] in fact there is *no* real explicit relation

```
General model Gauss7:
     f(x) =
     a1*exp(-((x-b1)/c1)^2) + a2*exp(-((x-b2)/c2)^2) +  a3*exp(-((x-b3)/c3)^2) +
     a4*exp(-((x-b4)/c4)^2) + a5*exp(-((x-b5)/c5)^2) + a6*exp(-((x-b6)/c6)^2) +
     a7*exp(-((x-b7)/c7)^2)

Coefficients (95% confidence bounds):
       a1 =      0.00334689            a4 =     -0.153761
       b1 =      3.09006               b4 =      0.509362
       c1 =      0.541368              c4 =      3.27624
       a2 =      0.000991008           a5 =      0.0101197
       b2 =      0.581446              b5 =      1.08525
       c2 =     -8.35204               c5 =      0.315601
       a3 =      1.57475               a6 =     -0.0296704
       b3 =     -5.55257               b6 =      1.95521
       c3 =      1.65756               c6 =      1.47912
                                       a7 =      0.474416
Goodness of fit:                       b7 =     -1.57139
SSE: 0.0001433                         c7 =      3.93275
R-square: 0.9989
Adjusted R-square: 0.9985
RMSE: 0.001533
```

The results (see the Goodness of fit indicators) comply perfectly with the second criterion we set up. Another point is to ensure the monotony. Taking a look at the Figure 18 we encounter an unwanted issue (the curve is not monotonous in the interval $x \in < 0, 0.461)$ and mainly KMV does not attribute $EDF >$ 20% which would be the case in the area close to $DD = 0$) that must be solved by imposing a condition

$$g(x) = \begin{cases} 0.2 & \text{for } x < 0.461 \\ f(x) & \text{for } x \geq 0.461 \end{cases} \tag{11.3}$$

and applying the condition we fulfil also the first criterion (secured real shape of the mapping curve). Note that in our case

$$\lim_{x \to \infty} f(x) = 0, \tag{11.4}$$

hence no conditioning for $x \to \infty$ is necessary. The algorithm $EDF(x) = g(x) = EDF(DD) = g(DD)$ is finally the desired explicit formula for $DD \to EDF$, where $EDF(DD) = g(x)$ given by the formula (11.3). The last remaining point is to put the

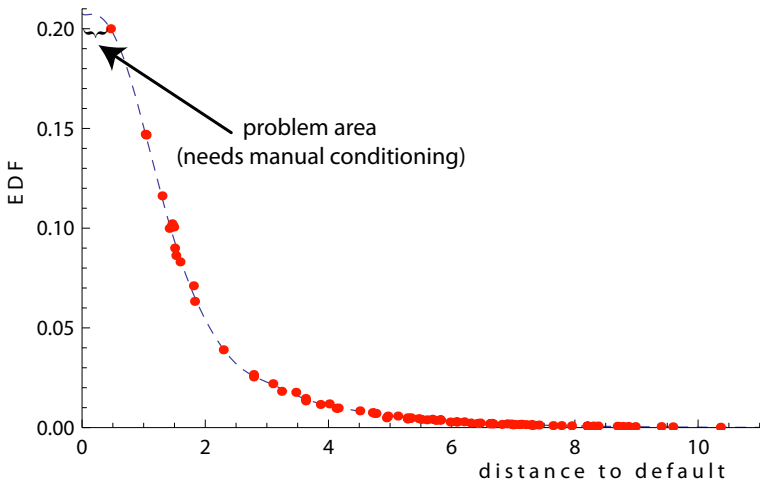

Figure 18: *New DD to EDF mapping algorithm: Gaussian fitting and monotony problem*

newly derived $DD \rightarrow EDF$ algorithm into practice, recalibrate the created model and perform the simulation. Taking a brief look at the relations $DD \rightarrow EDF$ in Figure 18 and comparing with our initial data set represented by Figure 12 we should expect much lower values of portfolio since the new data set is more critical and appoints to the same DD value approximately higher riskiness (higher EDF) that decreases portfolio value via the risk-adjusted discounting process.

We ignite once again 50 000 simulations and receive new results for the recalibrated model. Figure 19 depicts a comparison of KMV model with the previously used $DD \rightarrow EDF$ algorithm that we applied in Section 9 dedicated to practical demonstration of KMV model and newly calibrated model that is using $DD \rightarrow EDF$ mapping algorithm derived in this Appendix. Note that this time we apply the most sensitive data filter so that the charts are displayed in the most detailed way.

Apparently our initial rough hypothesis regarding the lower average portfolio value with new set of EDFs has been thus verified since the mean value has indeed decreased. Even though the true $DD \rightarrow EDF$ data are private property of KMV we at least can see the impact of change in EDFs. Regarding exact numeric data see Table 19. The new

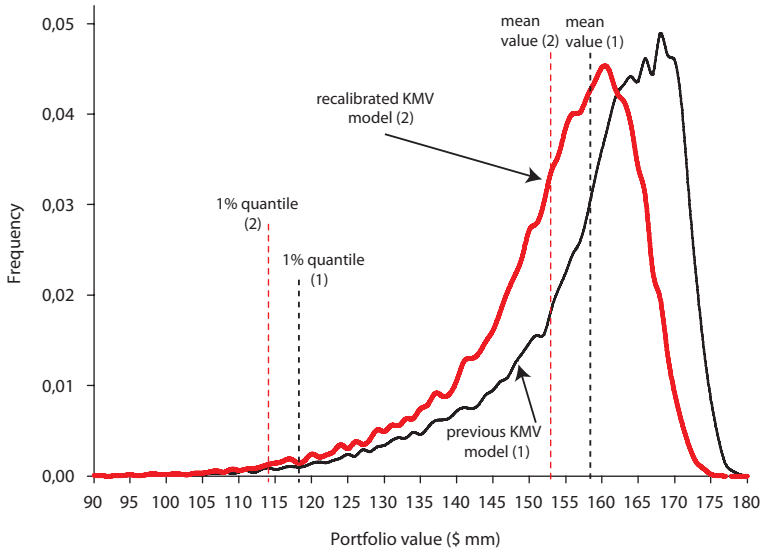

Figure 19: *Recalibrated (red) and original (black) KMV model*

	Mean	Min	Max	StDev	1% quantile	99% quantile	Unexp. L
New model	152.949	85.403	175.152	11.828	114.572	170.245	38.377
Old model	158.598	87.656	178.609	12.091	118.771	174.506	39.827

Table 19: *Results: recalibrated and old KMV model ($ mm)*

model is definitely more sceptical about the future. Besides significantly lower mean value also remarkably lower are both 1% quantile and 99% quantile and what is more both of the bounds the absolute Min and Max attained values also dropped. On the other hand the scepticism has resulted into a slightly lower unexpected loss and standard deviation.

Despite the fact that we can hardly assess which of those two models is closer to reality we may be sure that KMV stands and falls on a perfect calibration as such a small adjustment in modeling methodology has implied noticeable changes in final results.

Appendix: Moody's rating to KMV's EDF translation

In this Appendix our intention is to demonstrate how KMV's *EDF* is related to general rating grades (i.e. Moody's in this sample) as extra supplement to KMV. The main character of rating to *EDF* translation stems from the relation of, say, *implied* transition matrix of KMV model and of translation matrix created by rating agency. Table 20 depicts explicitly relation mentioned above. Note that KMV model usually provides clients with two types of probabilities with respect to a time horizon. See the Moody's *PD* is closer to the KMV's 5-year *PD* rather than to 1-year *PD*. It just underscores the assumption about TTC rating of rating agencies contra PIT of KMV. Also note that for the central rating categories (e.g. *A*) KMV argues (in the 5-year columns) that the *PD* is in fact higher than in the case of rating-based *PD*. The table in general nicely summarizes main differences between the sensitivity of KMV model and rating-based approach.

	Moody's		KMV (EDF 1 year)		KMV (EDF 5 years)	
	Low	High	Low	High	Low	High
Aaa	0.00%	0.27%	0.00%	0.019%	0.00%	0.28%
Aa1	0.28%	0.39%	0.020%	0.031%	0.29%	0.38%
Aa2	0.40%	0.49%	0.032%	0.051%	0.39%	0.52%
Aa3	0.50%	0.57%	0.052%	0.086%	0.53%	0.70%
A1	0.58%	0.60%	0.087%	0.143%	0.71%	0.86%
A2	0.61%	0.68%	0.144%	0.179%	0.87%	1.00%
A3	0.69%	1.04%	0.180%	0.223%	1.01%	1.35%
Baa1	1.05%	1.38%	0.224%	0.279%	1.36%	2.00%
Baa2	1.39%	2.08%	0.280%	0.429%	2.01%	2.97%
Baa3	2.09%	4.34%	0.43%	0.66%	2.98%	4.37%
Ba1	4.35%	6.55%	0.67%	1.10%	4.38%	5.90%
Ba2	6.56%	10.20%	1.11%	1.65%	5.91%	7.66%
Ba3	10.21%	14.77%	1.66%	2.48%	7.67%	9.96%
B1	14.78%	17.49%	2.49%	3.71%	9.97%	12.95%
B2	17.50%	21.51%	3.72%	5.57%	12.96%	16.84%
B3	21.52%	26.00%	5.58%	8.35%	16.85%	21.89%
Caa1/C	>26.00%		>8.35%		>21.89%	

Table 20: *Moody's rating to KMV's EDF conversion. (Source: (RMA, 2009))*

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