Stress-Testing Credit Risk

Transition Matrices

Version: 1.0

Date: 24.2.2014

February 2014

REVISIONS

Version	*Datum*	*Author*	*Comments*
0.01	18.12.2013	František Kalibán	Draft version
0.1	3.2.2014	František Kalibán	Example results
1.0	24.2.2014	Rostislav Černý	Revisions

Abstract

This paper describes purposes and principles of stress testing used in credit risk. One particular issue is stressing transition matrices among pools, for instance Probability of Default (PD) pools or Loss Given Default (LGD) pools. A possible approach to handle this problem is based on factor model assumed by the Basel Committee on Banking Supervision (BCBS), which is known as Merton or Vasicek model. In this paper, the model is generalized for transitions to more states than only the Default state, thus the theory can be used on transition matrices.

The theory is applied to the example of hypothetical mortgage loan portfolio, where the practical properties of Matrix stressing are shown.

1 Introduction

Recently, much attention is paid to so-called stress-testing, although the exact definition has never been stated. The framework is clear; stress-test should assess the ability of financial institution to handle stress situations. These situations occur naturally (at period of economic downturn) or suddenly (default of major bank, mortgage crisis, …), while the term stress-testing is more often used with handling the sudden events. The bank (or any other financial institution assessing the risk) should be prepared for such situations and have developed steps to do in case of their occurrence.

Stress testing can be considered as a risk management tool for evaluating unexpected risks. The regulators require the banks to hold a specified amount of capital, which is based on Vasicek formula (see Section 2.1). The formula itself is a form of a stress-test, where we take the worst 99.9% quantile of systematic factor affecting the PD. Thus we evaluate the unexpected loss for some unexpected but plausible situation.

According to BCBS, every bank must perform stress tests (see [1], Section III.H.4., paragraph 5):

"A bank must have in place sound stress-testing processes for use in the assessment of capital adequacy. These stress measures must be compared against the measure of expected positive exposure and considered by the bank as part of its internal capital adequacy assessment process. Stress-testing must also involve identifying possible events or future changes in economic conditions that could have unfavorable effects on a firm's credit exposures and assessment of the firm's ability to withstand such changes. Examples of scenarios that could be used are

(i) economic or industry downturns;
(ii) market-risk events; and
(iii) liquidity conditions.

In addition to the more general tests described above, the bank must perform a credit risk stress test to assess the effect of certain specific conditions on its IRB regulatory capital requirements…"

There are many methods to stress the regulatory capital requirement or economic capital. We can stress risk drivers (PD, LGD, EAD) independently or jointly, we can base the stress test on historical data or hypothetical scenarios, the stress can emerge from particular clients or the portfolio as a whole and there are other points of view we can distinguish.

2 Stressing probability matrix

One of the possible approaches is stressing the migration matrices between pools for a particular risk driver. In this paper we will consider hypothetical PD pools and stress probabilities of transition among them. The transition matrix is the stochastic matrix, i.e. each element must fulfill the condition $p_{ij} \geq 0$. Moreover the sum of each row must be equal one, for n pools

$$\sum_{j=1}^{n} p_{ij} = 1.$$

It means the stressing cannot be so straightforward, because e.g. taking some specific stress quantiles will lead to non-stochastic matrix. In order to retain stochastic matrix properties we can use several methods as Engelmann proposed in [4]. These are

- rescaling the migration probabilities,
- rescaling the generator matrix,
- shifting the migration probabilities.

These methods have some restrictions, e.g. the generator matrix does not have to exist and shifting or rescaling the migration probabilities needs some arbitrary selection of parameters.

In this paper we propose different method based on generalization of Vasicek model, which is consistent with Basel II formula. Consideration of systematic factor included in the transition probabilities will lead to possibility of stressing the matrix through the systematic factor.

2.1 Vasicek one-factor model

The Basel II formula for capital requirement is expressed as follows

(1) $$K = LGD \cdot \left[N\left(\frac{N^{-1}(PD) + \sqrt{\rho} \cdot N^{-1}(0.999)}{\sqrt{1-\rho}} \right) - PD \right],$$

where N is the cumulative standardized normal distribution function, N^{-1} its inverse, ρ is the correlation set up by the regulator (15% for mortgage loans, 4% for revolving loans, and somewhere between depending on PD for other retail loans). It will be useful to recall the principle of the formula that was firstly discovered by Vasicek [3].

The formula is formed by two terms, the unstressed LGD and stressed PD (unexpected part). The unexpected part of PD is often called unexpected default rate:

(2) $$UDR = N\left(\frac{N^{-1}(PD) + \sqrt{\rho} \cdot N^{-1}(0.999)}{\sqrt{1-\rho}} \right).$$

For a client i let T_i be the time to default on a client's debts. It is assumed that everyone will default once and as the time of the future event is unknown at present the time $T_i < \infty$ is a random variable. If Q_i is the cumulative probability distribution of T_i then it can be easily verified that the transformed variable $X_i = N^{-1}[Q_i(T_i)]$ is standardized normal (mean 0, standard deviation 1). The advantage is that after the transformation we can take the assumption that the variables are multivariate normal and given their mutual correlation ρ properties of normal variables can be used to obtain an analytic result. This approach is called the Gaussian copula model. The following one-factor model is used

(3) $X_i = \sqrt{\rho} * M + \sqrt{1-\rho} \cdot Z_i$

where M captures the systematic factor and Z_i the client specific. All Z_i's and M have independent standard normal distributions.

The one-year probability of default PD of the client i can be expressed as

(4)
$$P(T_i \le 1) = P(X_i \le N^{-1}[Q(1)]) = P\left(\sqrt{\rho} \cdot M + \sqrt{1-\rho} \cdot Z_i \le N^{-1}[Q(1)]\right)$$
$$= P\left(Z_i \le \frac{N^{-1}[Q(1)] - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}}\right) = N\left(\frac{N^{-1}[Q(1)] - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}}\right).$$

The next step is to consider M as the systematic driver of portfolio default rates. The model can be used for a simulation as follows: first generate randomly the value of an M from a standardized normal distribution and then independently all Z_i's. If the portfolio is large enough then the simulated default rate on the portfolio will be given by the formula above. If M is large the simulated default rate will be low, if M is smaller then the portfolio default rate will be higher. For a given probability level x the critical value of M is given by the quantile $N^{-1}(x)$. When M is replaced by $N^{-1}(x)$ and $Q(1)$ by the given average PD we get exactly the regulatory formula (2).

There is another possible interpretation of Vasicek model. The continuous non-observable variable X_i may be interpreted as the logarithmic return on an obligor's assets. A threshold-value model is assumed for the relationship between the return on assets and the default event D. Default is equivalent to the return on assets falling below the threshold c, i.e., $X_i < c, D_i = 1$.

2.2 Generalization of Vasicek model

We want to use Vasicek model in terms of multiple PD pools. Consider set of PD pools $S = \{1, 2, \ldots, n, d\}$ where n is number of non-default pools and d is the default state. Each year we can observe number of clients in particular pools. Let us transform the probability of default $P(T_i \leq 1)$ to one-year horizon transition among pools. Assume annual observations and that the debtor will remain in default state up to the observation time in case (s)he defaults but also up to one-year horizon, which is consistent with Basel II definition of default. Then for i-th client at time t

$$P_t(T_i \leq t+1) = P\left(Y_{t+1}^i = d\right),$$

where Y_t^i is the state which the i-th client belongs into at time t, $P_t(.)$ is the probability at time t. In case of no default at time $t+1$, the value of Y_{t+1}^i will belong to set $\{1, 2, \ldots, n\}$ and the probabilities of each of the options are given by the client credit quality grade.

Let's get back to formula (3). The random variable M indicates hidden systematic risk factor, e.g. macroeconomic situation. This variable affects all the clients in portfolio. The random variables Z_i, which denote obligor-specific idiosyncratic effect are independent and also unobservable. We can observe the resulting combined effect X_i, where the factor ρ determines the strength of common systematic factor.

In terms of multiple pools, symbol X_i stands for credit quality grade of i-th client. If the grade is lower at time t, the client will move to worse PD pool up to time $t+1$ (or default state, which can be considered as the worst PD pool), higher grade means transition to better PD pool. Thus the realization of variable X_i at time t affects the state Y_t^i in following way:

$$\left(Y_{t+1}^i = k\right) \Leftrightarrow \left(X_i \in \left[{}^jB_{k+1}, {}^jB_k\right]\right), \qquad j, k \in S,$$

where ${}^jB_{k+1}$ and ${}^jB_k$ are some thresholds for client leaving state j. Assume the process generating the systematic factor M is Markovian, as well as the process generating the idiosyncratic factor Z_i. Then for each client i the series Y_t^i form a homogenous Markovian process.

The goal is to set the thresholds ${}^jB_2, \ldots, {}^jB_n, {}^jB_d$ for variables X_i corresponding to clients leaving state j (for each $j \in S$). If realization of this variable x_i falls e.g. between thresholds ${}^jB_3$ and ${}^jB_2$ then the client i will move from pool j to pool 2. If the value of x_i is even higher than

${}^{j}B_2$, which is the highest threshold, the client will move to pool 1. The requirement is to retain the migration probabilities according to transition matrix of process $\{Y_t\}$ which we can observe.

2.3 Estimation

Transition matrix of $\{Y_t\}$, which can be observed from data is

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \dots & n & d \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} & p_{1d} \\ p_{21} & p_{22} & \cdots & p_{2n} & p_{2d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} & p_{nd} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{array},$$

and we want to set the default threshold ${}^{j}B_d$, for $j = 1, \dots, n$ to fulfill the equation

$$P(X_i < {}^{j}B_d) = p_{jd}.$$

If the client quality grade is even lower than ${}^{j}B_d$, the client leaving state j defaults. Since $X_i \sim N(0,1)$, the solution is ${}^{j}B_d = N^{-1}(p_{jd})$, where $N^{-1}(x)$ is the inverse of $N(0,1)$ CDF. Generally, the threshold ${}^{j}T_k$ for $k = 2, \dots, n$ must fulfill the equation

$$P({}^{j}B_{k+1} \le X_i < {}^{j}B_k) = p_{jk},$$

thus

$$P(X_i < {}^{j}B_k) = p_{jk} + p_{jk+1} + \cdots + p_{jd},$$

which has the solution ${}^{j}B_k = N^{-1}(p_{jk} + p_{jk+1} + \cdots + p_{jd})$. In terms of Z_i, we obtain

$$\begin{aligned} p_{jk} = P({}^{j}B_{k+1} \le X_i < {}^{j}B_k) &= P({}^{j}B_{k+1} \le \sqrt{\rho} \cdot M + \sqrt{1-\rho} \cdot Z_i < {}^{j}B_k) \\ &= P\left(\frac{{}^{j}B_{k+1} - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}} \le Z_i < \frac{{}^{j}B_k - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}}\right) \\ &= N\left(\frac{{}^{j}B_k - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}}\right) - N\left(\frac{{}^{j}B_{k+1} - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}}\right), \end{aligned}$$

Where $N(x)$ is CDF of $N(0,1)$ distribution. For $k = d$, following the same procedure, we obtain $p_{jd} = N\left(\frac{{}^{j}B_d - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}}\right)$ and for $k = 1$, we have

$$p_{j1} = P(X_i \ge {}^{j}B_2) = 1 - P(X_i < {}^{j}B_2) = \cdots = 1 - N\left(\frac{{}^{j}B_2 - \sqrt{\rho} \cdot M}{\sqrt{1-\rho}}\right).$$

Conditional probability for $M = m$ is

$$P\left({}^{j}B_{k+1} \leq {}^{j}X_i < {}^{j}B_k | M = m\right) = \cdots = N\left(\frac{{}^{j}B_k - \sqrt{\rho} \cdot m}{\sqrt{1-\rho}}\right) - N\left(\frac{{}^{j}B_{k+1} - \sqrt{\rho} \cdot m}{\sqrt{1-\rho}}\right) = {}^{stress}p_{jk}, \text{ for } k = 2, \ldots n,$$

where $n + 1 = d$ and similarly ${}^{stress}p_{j1} = 1 - N\left(\frac{{}^{j}B_2 - \sqrt{\rho} \cdot m}{\sqrt{1-\rho}}\right)$ and ${}^{stress}p_{jd} = N\left(\frac{{}^{j}B_d - \sqrt{\rho} \cdot m}{\sqrt{1-\rho}}\right)$.

As a stress factor, we can use $m = N^{-1}(0.001)$, which is consistent with stress scenario used under Basel II directive[1].

The construction of stress transition matrix from ${}^{stress}p_{jk}$ is obvious. Such matrix has the properties of transition probability matrix, i.e. row sums equal one and all of the elements lie between zero and one.

[1] The only difference is taking the lower quantile instead of the upper one due to the grade downturn is caused by lowering the value of ${}^{j}X_i$.

3 Case study

Consider hypothetical mortgage portfolio of clients belonging to 5 PD pools or default state (6th pool). In these pools, we have the number of observations at time T (the last moment we can observe the data) given by vector $\boldsymbol{n}_T$. Values of vector $\boldsymbol{n}'_T$ are given in following table:

Pool	1 (best)	2	3	4	5 (worst)	Default
# observations	13 882	9 075	3 740	3 117	2 886	1 467

Table 1: Number of observations at time T

We can observe numbers of clients at time $1, \ldots, T$, e.g. $T = 7$ for 7 years long time series. From these observations, we can obtain one-year transition matrix among pools $\{p_{ij}\}$ simply as a ratio of number of clients in pool j at time $t + 1$ divided by number of all clients in pool i at time t. The resulting matrix can look like this:

(6)
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ Default \end{array} \quad \boldsymbol{P}_{base} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & Default \end{array} \\ \begin{pmatrix} 0.92 & 0.01 & 0.01 & 0.04 & 0.01 & 0.00 \\ 0.11 & 0.78 & 0.01 & 0.07 & 0.02 & 0.01 \\ 0.47 & 0.17 & 0.65 & 0.03 & 0.09 & 0.01 \\ 0.22 & 0.20 & 0.05 & 0.37 & 0.14 & 0.02 \\ 0.04 & 0.06 & 0.11 & 0.13 & 0.51 & 0.14 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array}.$$

The scenario for time $T + 1$ we get using transition probabilities according to this matrix is called baseline scenario, i.e. $\boldsymbol{n}'_{T+1} = \boldsymbol{n}'_T \boldsymbol{P}_{base}$. This is the progress we expect the portfolio to follow.

Using the stress procedure described above, i.e. taking the stress factor $m = N^{-1}(0.001)$ and correlation given by the regulator (for mortgages, $\rho = 15\%$), we get stressed transition matrix

(7)
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ Default \end{array} \quad \boldsymbol{P}_{stress} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & Default \end{array} \\ \begin{pmatrix} 0.60 & 0.04 & 0.04 & 0.19 & 0.09 & 0.05 \\ 0.00 & 052 & 0.02 & 0.24 & 0.15 & 0.08 \\ 0.00 & 0.02 & 0.46 & 0.08 & 0.34 & 0.11 \\ 0.02 & 0.05 & 0.02 & 0.33 & 0.40 & 0.18 \\ 0.00 & 0.00 & 0.01 & 0.03 & 0.41 & 0.55 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array}.$$

Similarly, we can obtain stressed number of observations at time $T + 1$ as $\boldsymbol{n}'_{T+1} = \boldsymbol{n}'_T \boldsymbol{P}_{stress}$. Numbers of observations at time $T + 1$ are stated in Table 2 and Figure 1.

Pool	1 (best)	2	3	4	5 (worst)	Default
T	13 882	9 075	3 740	3 117	2 886	1 467
Baseline T+1	14 778	8 748	3 150	2 838	2 582	2 071
Stressed T+1	8 361	5 452	2 526	6 146	6 301	5 381

Table 2: Numbers of observations after transition, i.e. at time T+1

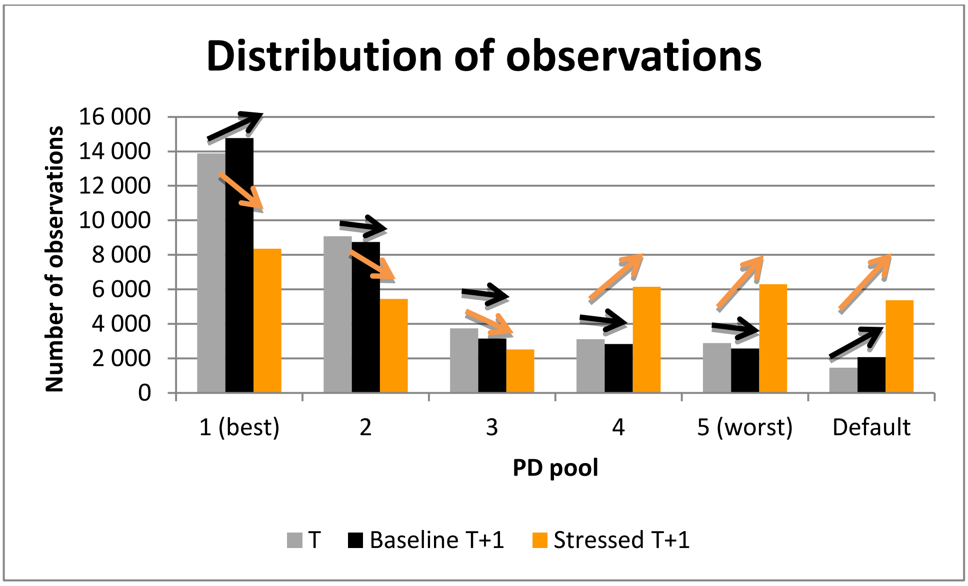

Figure 1: Distribution of observations

We can see the numbers of observations remain almost the same by taking the baseline scenario transitions. The last line shows that the number of clients decreases in the pools with "better" clients (1-3). On the contrary, the numbers of clients increase in worse PD pools (4, 5) and in "Default pool".

Comparing the last column of transition matrix, we can see the probabilities of default in particular pools increased significantly. Let's assume $LGD = 100\%$ and EA$D = 1$ and evaluate the capital requirement according to Basel II formula based on Vasicek model as well as the expected loss on the portfolio. The capital requirement for existing debtors at time $T + 1$ is evaluated according to formula (1). Under the baseline scenario is $K = 9.64\%$. Similarly, the expected loss given by $EL = LGD \times PD \times EAD$ gives the baseline expected loss $EL = 7.67\%$. The estimation of stressed capital requirement and stressed expected loss can be done for following approaches:

Through the Cycle (TTC)

In case we want to assess K in terms of TTC approach, we should use the long-term PDs. These estimates can be obtained by PD model or as probabilities of transition of baseline scenario. In our case, we can find these values in the last column of matrix (6) $\boldsymbol{P}_{base}$. For the particular example, this approach results in $K = 15.92\%$ and $EL = 18.95\%$. The stress was made solely by stressing the numbers in PD pools.

Point in Time (PIT)

Another approach is to follow PIT methodology. Since we expect the PDs to be increased by the stress situation, we should increase them also in capital requirement calculation. Taking the stressed values from matrix (7) $\boldsymbol{P}_{stress}$, we get $K = 34.41\%$ for the stressed scenario and $EL = 32.36\%$. The stress was made by stressing the numbers in PD pools and PDs theirselves.

4 Conclusion

As we could have seen in previous section, different approaches lead to different results. Approach based on TTC methodology lead to mild stressing which corresponds to degradation of number of good clients. PIT methodology leads to degradation of both the number of good clients and corresponding degradation of PDs across the PD pools. The amounts of capital[2] allocated by each of approaches are given by Figure 2 and Figure 3.

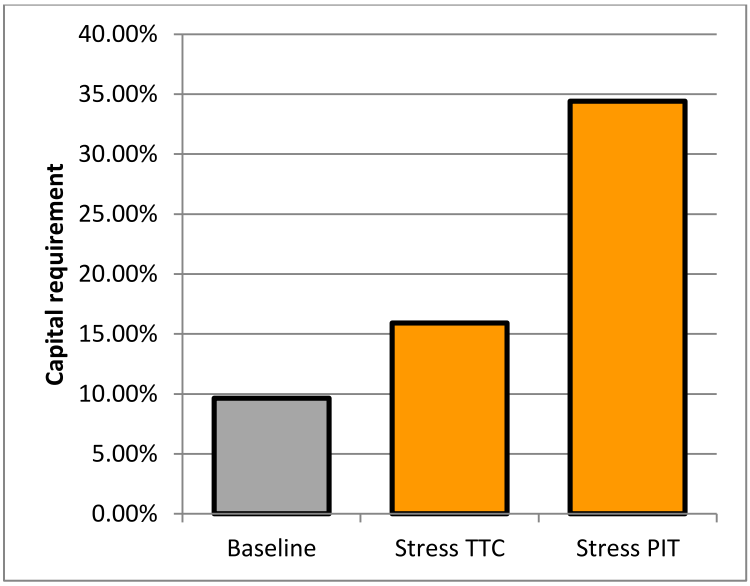

Figure 2: Capital requirement according to different stress approaches

[2] The capital requirement K is obviously computed on different amount of nondefaulted clients, since the number of defaults varies with the scenario

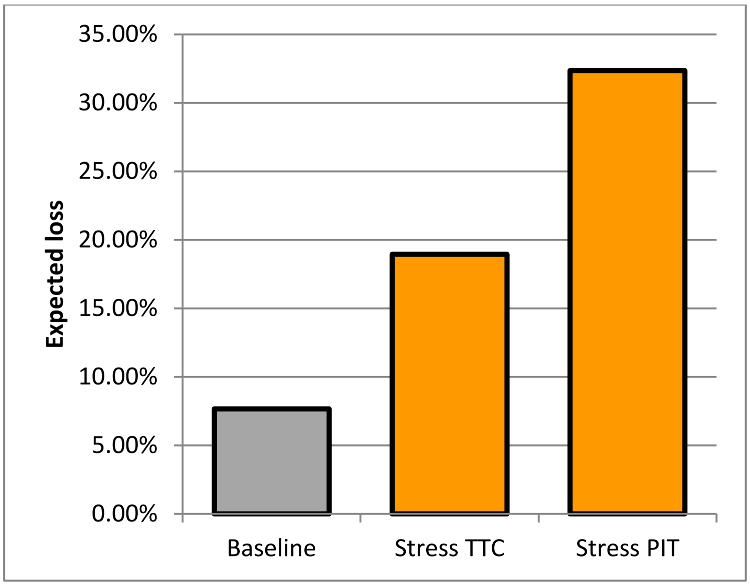

Figure 3: Expected loss according to different stress approaches

5 References

[1] Rösch, Daniel, and Harald Scheule. "Stress-testing credit risk parameters: an application to retail loan portfolios." *Journal of Risk Model Validation* 1.1 (2007): 55-75.

[2] Basel Committee on Banking Supervision, and Bank for International Settlements. International convergence of capital measurement and capital standards: a revised framework. Basel Committee on Banking Supervision, 2004.

[3] Vasicek, Oldrich. "Probability of loss on loan portfolio." KMV Corporation 12.6 (1987).

[4] Engelmann Bernd. "Macroeconomic Stress Testing" http://www.garp.org/media/1153030/macroeconomicstresstesting_berndengelmann_042413.pdf