Interest Rate Swap Credit Value Adjustment

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Abstract

The credit value adjustment (CVA) of OTC derivatives is an important part of the Basel III credit risk capital requirements and current accounting rules. Its calculation is not an easy task - not only it is necessary to model the future value of the derivative, but also the probability of default of a counterparty. Another complication arises in the calculation incorporating the wrong-way risk, i.e. the negative dependence between the underlying asset and the default time. A semi-analytical CVA formula simplifying the interest rate swap (IRS) valuation with counterparty credit risk (CCR) including the wrong-way risk is derived and analyzed in the paper. The formula is based on the fact that the CVA of an IRS can be expressed by the swaption price. The link between the interest rates and the default time is represented by a Gaussian copula with constant correlation coefficient. Finally, the results of the semi-analytical approach are compared with results of a complex simulation study.

JEL classification code: C63, G12, G13, G32

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1 Introduction

Although the principle of the CVA is known and applied by some banks more than twenty years, it is coming to the forefront nowadays with the new Basel III regulation. Basel III regulation became effective in 2010 but its implementation has been phased into years 2013-2019.

In this paper we consider CCR for IRS in presence of constant correlation between the default time and interest rates. In particular we present a model-free linear dependence formula for CVA, i.e. formula independent of the hazard rate model and underlying asset model, and a semi-analytical formula for IRS CVA. From [7] we know that a swap price including CCR without wrong-way risk can be calculated as a swaption price weighted by the risk-neutral probability of default. The IRS CVA formula is an extension of the risky option price from [5] to the IRS price with CCR including wrong-way risk.

We show that CCR has a relevant impact on the IRS prices and that the wrong-way risk has a relevant impact on the CVA. We analyze the results of the semi-analytical formula and simulation study provided in [2]. In [2] Brigo and Pallavicini assume G2++ model for interest rates development and CIR++ for hazard rate development which requires quite complex modeling. It turns out that for zero correlation between interest rates and default time (resp. hazard rate) the results of the semi-analytical formula and the simulation approach are similar. For non-zero correlation the results vary which is caused by different correlation calculation. The correlation in [2] represents dependence between the instantaneous differences of the interest rates and default intensity, i.e. instantaneous correlation, while in the semi-analytical formula the correlation is between absolute values of the default time and interest rates. In our view, higher interest rates economically cause more defaults over time and so the correlation between the level interest rates and the time to default is economically better interpretted than a correlation between instantenous differences. We discuss the results, findings and possible challanges in more detail in the conclusions.

2 Unilateral CVA valuation

Denote the risk-neutral value of the discounted financial derivative at time t expiring at time T as $V(t,T)$ and the discounted value of the derivative including CCR as $V^\star(t,T)$.

Theorem 1 *Assume τ is the default time of the counteparty, Q is the risk-neutral measure and RR is a constant recovery rate on any defaulted amount then*

$$V^\star(t,T) = V(t,T) - \mathrm{CVA}(t,T), \tag{2.1}$$

where

$$\mathrm{CVA}(t,T) = \mathbb{E}^Q\left[(1-\mathrm{RR})\mathbb{1}_{[\tau\le T]}\cdot\max(V(\tau,T),0)\right].$$

Proof. First, let us denote $V(t,T)^+ = \max(V(t,T),0)$, $V(t,T)^- = -\max(-V(t,T),0)$ and $V(t,\tau)$ as the value of cashlow from t to τ. Then

$$\begin{aligned} V^\star(t,T) &= \mathbb{E}^Q\left[\mathbb{1}_{[\tau>T]}V(t,T) + \mathbb{1}_{[\tau\le T]}V(t,\tau) \right. \\ &\quad \left. +\mathbb{1}_{[\tau\le T]}(\mathrm{RR}\cdot V(\tau,T)^+ + V(\tau,T)^-)\right] \\ &= \mathbb{E}^Q\left[\mathbb{1}_{[\tau>T]}V(t,T) + \mathbb{1}_{[\tau\le T]}V(t,\tau) \right. \\ &\quad \left. +\mathbb{1}_{[\tau\le T]}((\mathrm{RR}-1)V(\tau,T)^+ + V(\tau,T))\right] \\ &= V(t,T) - \mathbb{E}^Q\left[(1-\mathrm{RR})\mathbb{1}_{[\tau\le \mathrm{T}]}\mathrm{V}(\tau,\mathrm{T})^+\right]. \end{aligned}$$

□

RR stands for the recovery rate of the derivative and $(1-\mathrm{RR}) \equiv \mathrm{LGD}$ is the loss given default (in percent of the exposure). So the CVA of the derivative value is the expectation of the irrecoverable part of the exposure in the risk-neutral valuation.

The expectation may be rewritten into the following form

$$\mathbb{E}^Q\left[\mathrm{LGD}\cdot\mathbb{1}_{[\tau\le T]}V(\tau,T)^+\right] = -\mathbb{E}^Q\int_t^T \mathrm{LGD}\cdot V(u,T)^+\mathrm{dS(u)} \tag{2.2}$$

where $S(t) = P(\tau > t) = 1 - P(\tau \le t)$ is the survival function of the counterparty. In case that the discounting is not included in $V(t,T)$, the above formula should look like

$$\mathbb{E}^Q\left[\mathrm{LGD}\cdot\mathbb{1}_{[\tau\le T]}D(t,\tau)V(\tau,T)^+\right] = -\mathbb{E}^Q\int_t^T \mathrm{LGD}\cdot D(t,u)V(u,T)^+\mathrm{d}S(u), \tag{2.3}$$

where $D(t,T)$ is generally stochastic risk-neutral discount factor from the time T to time t.

3 Hazard rate

One of possible approaches to calculate the default probability is using the default intensity or hazard rate. It means probability of default occuring in "infinitesimaly small" time step. More rigorously, if F is the cummulative distribution function of τ then

$$\begin{aligned} P(t<\tau<t+\Delta t|\tau\ge t) &= \frac{P(t<\tau<t+\Delta t,\tau\ge t)}{P(\tau\ge t)} \\ &= \frac{F(t+\Delta t)-F(t)}{S(t)} = \frac{F(t+\Delta t)-F(t)}{S(t)\Delta t}\cdot\Delta t. \end{aligned}$$

For $\Delta t \to 0$ and F absolutely continuous

$$P(t < \tau < t + \Delta t | \tau \geq t) \to f(t)\mathrm{d}t \frac{1}{S(t)} = -S'(t)\mathrm{d}t \frac{1}{S(t)} = \lambda(t)\mathrm{d}t, \tag{3.1}$$

where $\lambda(t)$ is the hazard rate and f is the density function of τ regarding Lebesgue measure.

We can use the hazard rate to simplify CVA expression into the form

$$-\mathbb{E}^Q \int_t^T \mathrm{LGD} \cdot D(t,u)V(u,T)^+ \mathrm{d}S(u) = \mathbb{E}^Q \int_t^T \mathrm{LGD} \cdot D(t,u)V(u,T)^+ \lambda(u)S(u)\mathrm{d}u.$$

In particular we will use exponential distribution of the default time which is the most important distribution in the survival theory. The main property of the exponential distribution is its memoryless and constant hazard rate (intensity). Of course it is also possible to use other distributions, e.g. Weibull, Lognormal, Loglogistic (see [8]), or nonparametric approach which would be very difficult to implement for the lack of data.

4 Model-free linear dependence

Following expression of the CVA including wrong-way risk is just a statistical simplification which uses the CVA without wrong-way risk and the linear dependence in form of the correlation coefficient.

Let us assume that the dependence between the value of the underlying asset and the default time (wrong-way risk) is linear. Then the correlation coefficient can be used to calculate CVA. Model-free formula has following form

$$\begin{aligned} \mathbb{E}^Q \left[\mathrm{LGD} \cdot \mathbb{1}_{[\tau \leq T]} V(\tau,T)^+\right] &= \mathrm{LGD}\left(\mathrm{cov}\left[\mathbb{1}_{[\tau \leq T]}, V(\tau,T)^+\right]\right. \\ &\quad \left. + \mathbb{E}^Q\left[\mathbb{1}_{[\tau \leq T]}\right] \cdot \mathbb{E}^Q\left[V(\tau,T)^+\right]\right) \end{aligned}$$

assuming that LGD is constant or deterministic. Second part of the formula including LGD is the known CVA without wrong-way risk, so we have

$$\mathrm{CVA}_{\mathrm{wrong}}(t,T) = \mathrm{LGD} \cdot \mathrm{cov}\left[\mathbb{1}_{[\tau \leq T]}, V(\tau,T)^+\right] + \mathrm{CVA}(t,T), \tag{4.1}$$

The covariance of the exposure and the default time can be expressed by the correlation coefficient $\rho \in [-1,1]$ as

$$\mathrm{cov}\left[\mathbb{1}_{[\tau \leq T]}, V(\tau,T)^+\right] = \rho\sqrt{\mathrm{var}\left[\mathbb{1}_{[\tau \leq T]}\right] \mathrm{var}\left[V(\tau,T)^+\right]}. \tag{4.2}$$

The indicator of the default time has a Bernoulli distribution with probability p depending on the hazard rate [1]

$$\mathrm{var}\left[\mathbb{1}_{[\tau\leq T]}\right] = [1 - p(\lambda(T))]p(\lambda(T)). \tag{4.3}$$

Assuming exponential distribution of the default time with constant parameter, the variance can be expressed as

$$\mathrm{var}\left[\mathbb{1}_{[\tau\leq T]}\right] = e^{-\lambda T}(1 - e^{-\lambda T}).$$

The second part of the covariance depends on the underlying asset of the derivative so it cannot be easily simplified as the variance of the indicator function.

5 Risky Swaption Price

First, we will recall the well-known swaption price formula based on the Black's model and then we will focus on its extended risky version.

As noted in [9], payer swaption price at time 0 based on the Black's model with nominal value L, swaption strike rate s_K, forward swap rate s_0 of a swap starting at the excercise date T with fixed coupon rate s_T at times $T_1, \ldots, T_n$ and $\sigma^2 T$ the log-variance of s_T, is

$$V_{pay}(0, T, T_n) = X(0) \cdot L \cdot (s_0\Phi(d_1) - s_K\Phi(d_2))$$

where

$$\begin{aligned} d_1 &= \frac{\log(s_0/s_K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Swaption price is calculated as the discounted expectation of the payoff. But typical money market account discounting is replaced by another one, so called annuity, in the form

$$X(0) = \sum_{i=1}^{n} \delta_i P(0, T_i)$$

where $P(0, T_i)$ is the zero coupon bond value with maturity at T_i and δ_i is the time factor related to the period (T_{i-1}, T_i).

Theorem 2 *Suppose that in the risk-neutral world swap rate s_T development follows Black's model with a constant volatility $\sigma > 0$, i.e.*

$$\begin{aligned} \mathrm{d}s_t &= \sigma s_t \mathrm{d}Y, \\ s_T &= s_0 \exp\left\{-\sigma^2 T/2 + \sigma\sqrt{T}Y\right\}, \end{aligned} \tag{5.1}$$

[1] For exponential distribution with constant parameter λ the probability $p(\lambda(T)) = P(\mathbb{1}_{[\tau>T]} = 1) = e^{-\lambda T}$

where

$$Y = \sqrt{\rho}U + \sqrt{1-\rho}\varepsilon_1 \quad \rho \in [0,1].$$

The default time is defined as

$$\tau = S^{-1}(1-\Phi(Z)),$$

where $S(t) = e^{-ht}$, h *is a hazard rate and*

$$Z = \sqrt{\rho}U + \sqrt{1-\rho}\varepsilon_2.$$

U, ε_1 *and* ε_2 *are independent standard Gaussian random variables. Then the risky payer, resp. receiver, swaption price with strike rate* s_K, *no recovery and with the payoff function*

$$V(T,T,T_n) = \begin{cases} X(T)\cdot \mathbb{1}_{[\tau>T]}\left(s_T - s_K\right)^+ & \textit{for payer swaption} \\ X(T)\cdot \mathbb{1}_{[\tau>T]}\left(s_K - s_T\right)^+ & \textit{for receiver swaption} \end{cases}$$

is

$$V^\star_{pay}(0,T,T_n) = L\cdot X(0)\cdot(s_0\cdot A_1 - s_K\cdot A_2)$$

resp.

$$V^\star_{rec}(0,T,T_n) = L\cdot X(0)\cdot(s_K\cdot A_{-2} - s_0\cdot A_{-1}),$$

where

$$\begin{aligned}
A_{\pm 1} &= \int_{-\infty}^{\infty} \exp\left\{\sqrt{\rho}u\sigma\sqrt{T} - \rho\sigma^2 T/2\right\}\Phi\left(\pm\frac{d_1+\sqrt{\rho}u-\rho\sigma\sqrt{T}}{\sqrt{1-\rho}}\right)\cdot \\
&\quad \Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u, \\
A_{\pm 2} &= \int_{-\infty}^{\infty} \Phi\left(\pm\frac{d_2+\sqrt{\rho}u}{\sqrt{1-\rho}}\right)\Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u, \\
d_1 &= \frac{\log\left(A(0)/K\right) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \\
d_2 &= d_1 - \sigma\sqrt{T}, \\
\varphi(u) &= \frac{1}{\sqrt{2\pi}}e^{-u^2/2}, u\in\mathbb{R}.
\end{aligned}$$

Proof. The present price of the payer swaption is calculated as the discounted expectation of the future payoff, i.e.

$$V^{\star}_{pay}(0,T,T_n) = L \cdot X(0) \cdot \mathbb{E}^Q \left[\frac{V(T,T,T_n)}{X(T)} \right] = L \cdot X(0) \cdot \mathbb{E}^Q \left[\mathbb{1}_{[\tau > T]} (s_T - s_K)^+ \right].$$

The default time τ can be expressed by the standard Gaussian variables U and ε_1, so

$$\begin{aligned}
\mathbb{E}\left[(s_T - s_K)^+ \mathbb{1}_{[\tau>T]}\right] &= \mathbb{E}\left[\mathbb{E}\left[(s_T - s_K)^+ \mathbb{1}_{[\tau>T]} | U\right]\right] \\
&= \int_{-\infty}^{\infty} \mathbb{E}\left[(s_T - s_K)^+ \mathbb{1}_{[Z > \Phi^{-1}(1-S(T))]} | U = u\right] \varphi(u) \mathrm{d}u \\
&= \int_{-\infty}^{\infty} \mathbb{E}\left[(s_T - s_K)^+ \mathbb{1}_{\left[\varepsilon_2 > \frac{\Phi^{-1}(1-S(T)) - \sqrt{\rho} u}{\sqrt{1-\rho}}\right]} | U = u\right] \varphi(u) \mathrm{d}u.
\end{aligned}$$

The stochastic part of the swap rate s_T with given $U = u$ is equal ε_1. The independence of ε_1 and ε_2 implies

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathbb{E}\left[(s_T - s_K)^+ \mathbb{1}_{\left[\varepsilon_2 > \frac{\Phi^{-1}(1-S(T)) - \sqrt{\rho} u}{\sqrt{1-\rho}}\right]} | U = u\right] \varphi(u) \mathrm{d}u = \\
&= \int_{-\infty}^{\infty} \mathbb{E}\left[(s_T - s_K)^+ | U = u\right] \mathbb{E}\left[\mathbb{1}_{\left[\varepsilon_2 > \frac{\Phi^{-1}(1-S(T)) - \sqrt{\rho} u}{\sqrt{1-\rho}}\right]} | U = u\right] \varphi(u) \mathrm{d}u \\
&= \int_{-\infty}^{\infty} \mathbb{E}\left[(s_T - s_K)^+ | U = u\right] \int_{-\infty}^{\infty} \mathbb{1}_{\left[x_2 > \frac{\Phi^{-1}(1-S(T)) - \sqrt{\rho} u}{\sqrt{1-\rho}}\right]} \varphi(x_2) \mathrm{d}x_2 \varphi(u) \mathrm{d}u \\
&= \int_{-\infty}^{\infty} \mathbb{E}\left[(s_T - s_K)^+ | U = u\right] \Phi\left(\frac{\sqrt{\rho} u - \Phi^{-1}(1 - S(T))}{\sqrt{1-\rho}} \right) \varphi(u) \mathrm{d}u \\
&= \int_{-\infty}^{\infty} \int_{\frac{\log s_K - m}{w}}^{\infty} (e^{x_1 w + m} - s_K) \varphi(x_1) \mathrm{d}x_1 \Phi\left(\frac{\sqrt{\rho} u - \Phi^{-1}(1 - S(T))}{\sqrt{1-\rho}} \right) \varphi(u) \mathrm{d}u,
\end{aligned}$$

where

$$\begin{aligned}
m &= \log(s_0) - \sigma^2 T/2 + \sqrt{\rho} u \sigma \sqrt{T}, \\
w &= \sqrt{1-\rho} \sigma \sqrt{T}.
\end{aligned}$$

Now we can separate the integral in two parts. First, we will integrate the part with the strike rate s_K

$$
\begin{aligned}
& -s_K \int_{-\infty}^{\infty} \int_{\frac{\log s_K - m}{w}}^{\infty} \varphi(x_1)\mathrm{d}x_1 \Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u = \\
= \; & -s_K \int_{-\infty}^{\infty} \Phi\left(\frac{m - \log K}{w}\right)\Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u = \\
= \; & -s_K \int_{-\infty}^{\infty} \Phi\left(\frac{\log(s_0/s_K) - \sigma^2 T/2 + \sqrt{\rho}u\sigma\sqrt{T}}{\sqrt{1-\rho}\sigma\sqrt{T}}\right)\Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u = \\
= \; & -s_K \int_{-\infty}^{\infty} \Phi\left(\frac{d_2 + \sqrt{\rho}u}{\sqrt{1-\rho}}\right)\Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u = -s_K \cdot A_2.
\end{aligned}
$$

then the part with the swap rate s_0

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \int_{\frac{\log s_K - m}{w}}^{\infty} e^{x_1 w + m}\varphi(x_1)\mathrm{d}x_1 \Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u = \\
= \; & \int_{-\infty}^{\infty} \int_{\frac{\log s_K - m}{w} - w}^{\infty} e^{m + w^2/2}\varphi(x_1)\mathrm{d}x_1 \Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u \\
= \; & \int_{-\infty}^{\infty} e^{m + w^2/2}\Phi\left(w - \frac{log\,(s_K) - m}{w}\right)\Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u \\
= \; & s_0 \int_{-\infty}^{\infty} \exp\left\{\sqrt{\rho}u\sigma\sqrt{T} - \rho\sigma^2 T/2\right\}\Phi\left(\frac{d_1 + \sqrt{\rho}u - \sigma\sqrt{T}\rho}{\sqrt{1-\rho}}\right) \cdot \\
& \Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1-S(T))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u \\
= \; & s_0 \cdot A_1.
\end{aligned}
$$

The proof of the receiver swaption price is analogous.

□

6 Interest Rate Swap CVA

The CVA of the interest rate swap (hereinafter "CVA_{IRS}") in case of no wrong-way risk can be expressed as a sum of interest rate swaption prices

weighted by survival probabilities, as noted in [7]. The formula presented in [4] is in form

$$\mathrm{CVA}_{IRS} \approx LGD \cdot \sum_{i=0}^{n-1} (S(T_i) - S(T_{i+1}))V(t, T_{i+1}, T). \tag{6.1}$$

where n is the number of swap payments, $V(t, T_{i+1}, T)$ is the present swaption price with option expiration at time T_{i+1} and swap maturity at time T.

Even in the presence of the wrong-way risk we can use the swaption price to evaluate the CVA_{IRS} with n swap payments $(0 = t = T_0 < T_1 < \cdots < T_n = T)$

$$\begin{aligned}
\mathrm{CVA}_{IRS}(T_i, T_{i+1}) &= X_i(t)\mathbb{E}^Q\left[\mathbb{1}_{[T_i \le \tau < T_{i+1}]} V(t, T_{i+1}, T)^+\right] \\
&= X_i(t)\mathbb{E}^Q\left[\mathbb{1}_{[\tau \ge T_i]} V(t, T_{i+1}, T)^+ - \mathbb{1}_{[\tau \ge T_{i+1}]} V(t, T_{i+1}, T)^+\right] \\
&= X_i(t)\mathbb{E}^Q\left[\mathbb{1}_{[\tau \ge T_i]} V(t, T_{i+1}, T)^+\right] - X_i(t)\mathbb{E}^Q\left[\mathbb{1}_{[\tau \ge T_{i+1}]} V(t, T_{i+1}, T)^+\right], \\
i = 0, \ldots, n-1,
\end{aligned}$$

where

$$X_i(t) = \sum_{j=i+1}^{n} \delta_j P(t, T_j).$$

In other words, the CVA_{IRS} from the time T_i up to time T_{i+1} is equal to the difference of the swaption prices, i.e. options (default part) on the future swap trades (derivative part).

The relation between the swaption price and CVA formula is following. If we put $V(t, T_{i+1}, T) = (s_{T_{i+1}} - s_K) \cdot X(T_{i+1})$ then

$$\mathbb{E}_X^{\mathbb{Q}}\left[\frac{\mathbb{1}_{[\tau > T_{i+1}]} V(T_{i+1}, T)^+}{X(T_{i+1})}\right] = \mathbb{E}_X^{\mathbb{Q}}\left[\mathbb{1}_{[\tau > T_{i+1}]} (s_{T_{i+1}} - s_K)^+\right],$$

where $\mathbb{E}_X^{\mathbb{Q}}[\,\cdot\,]$ is the expectation regarding annuity numeraire.

Finally, the $\mathrm{CVA}_{IRS}(t, T)$ is calculated as

$$\mathrm{CVA}_{IRS}(t, T) = \sum_{i=0}^{n-1} \mathrm{CVA}_{IRS}(T_i, T_{i+1})$$

which is the sum of the differences of the swaption prices in various time periods.

Price of such swaptions can be calculated using risky swaption formula presented earlier. Then the IRS price including CCR can be simply calculated using the semi-analytical forumla for the CVA_{IRS}

$$\mathrm{CVA}_{IRS}(t, T) = L \sum_{i=0}^{n-1} X_i(t) \left[s_{t,i}(A_{1,i} - B_{1,i}) - s_K(A_{2,i} - B_{2,i})\right]$$

where

$$
\begin{aligned}
A_{1,i} &= \int_{-\infty}^{\infty} \exp\left\{\sqrt{\rho}u\sigma\sqrt{T_{i+1}} - \rho\sigma^2 T_{i+1}/2\right\}\Phi\left(\frac{d_{1,i} + \sqrt{\rho}u - \rho\sigma\sqrt{T_{i+1}})}{\sqrt{1-\rho}}\right) \cdot \\
&\quad \Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1 - S(T_i))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u, \\
B_{1,i} &= \int_{-\infty}^{\infty} \exp\left\{\sqrt{\rho}u\sigma\sqrt{T_{i+1}} - \rho\sigma^2 T_{i+1}/2\right\}\Phi\left(\frac{d_{1,i} + \sqrt{\rho}u - \rho\sigma\sqrt{T_{i+1}})}{\sqrt{1-\rho}}\right) \cdot \\
&\quad \Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1 - S(T_{i+1}))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u, \\
A_{2,i} &= \int_{-\infty}^{\infty} \Phi\left(\frac{d_{2,i} + \sqrt{\rho}u}{\sqrt{1-\rho}}\right)\Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1 - S(T_i))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u, \\
B_{2,i} &= \int_{-\infty}^{\infty} \Phi\left(\frac{d_{2,i} + \sqrt{\rho}u}{\sqrt{1-\rho}}\right)\Phi\left(\frac{\sqrt{\rho}u - \Phi^{-1}(1 - S(T_{i+1}))}{\sqrt{1-\rho}}\right)\varphi(u)\mathrm{d}u, \\
d_{1,i} &= \frac{\log\left(s_{t,i}/s_K\right) + \sigma^2 T_{i+1}/2}{\sigma\sqrt{T_{i+1}}}, \\
d_{2,i} &= d_{1,i} - \sigma\sqrt{T_{i+1}}.
\end{aligned}
$$

and the swap rate $s_{t,i}$ is the forward swap rate of the swap starting at time T_{i+1}.

7 Numerical study

We are going to price a plain-vanilla at-the-money 10Y IRS (with swap rate 4.05 %) on the EUR market where the fixed leg pays annually a 30E/360 strike rate and the floating leg pays semi-annually LIBOR. The recovery rate is equal zero, i.e. $LGD = 1$, and the volatility σ is equal 12 %. Remaining inputs of the model are the zero-bond spot rates which are shown in the Appendix.

We considered in the Theorem 2 positive correlation coeficient which means that the right-way risk is not included, only wrong-way risk, which is a conservative approach. The correlation coeficient describes the linear dependence between absolute values of the default time and the interest rate (not between their instantaneous changes). The following table shows the CVA of the IRS including CCR using semi-analytical formula for different correlation coefficients.

ρ	$\mathbf{CVA_{IRS}}$
0	0.343 %
0.1	0.394 %
0.3	0.501 %
0.5	0.619 %
0.7	0.751 %
0.9	0.914 %
1	1.028 %

Table 1: $\mathrm{CVA_{IRS}}$ values as a percentage of the notional amount with hazard rate equal 5 %

The impact of the wrong-way risk is certainly not negligible. In case of perfect correlation between interest rate and default time the wrong-way risk is about 0.7 % of the nominal value.

Previously presented CVA was related to the risk-neutral hazard rate equal 5 % which corresponds to worse rated countries or well-rated companies. A corporate risk-neutral hazard rate will be typically about 10 % (see Moody's rating B in [6]). The figure below illustrates the behavior of the CVA of the IRS with respect to correlation coefficient and also the hazard rate.

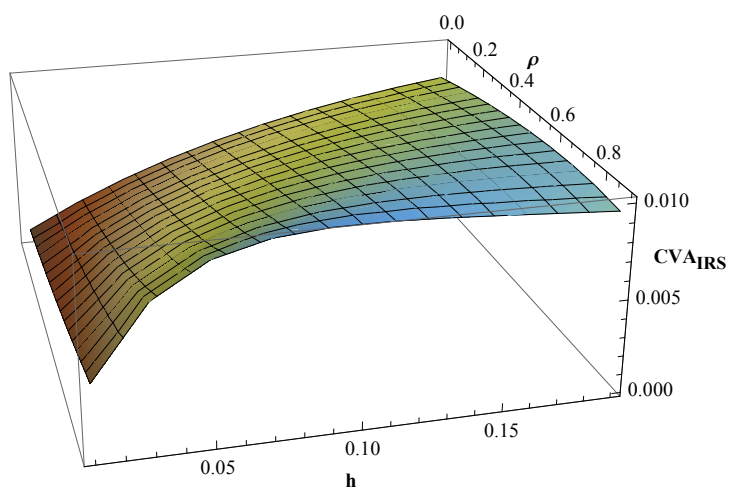

Figure 1: $\mathrm{CVA_{IRS}}$ values as a function of hazard rate h and correlation ρ

We may see that from a certain values of the hazard rate and correlation the CVA tends to go down. This is caused by the high probability of the early default, i.e. the IRS price will not change much during short time interval neither does the exposure.

8 Comparison with simulation study

We compared the prices of IRS including CCR calculated by the semi-analytical approach presented in previous section with the simulation study provided in [2].

In [2] the swap price development is described by the two-factor G2++ model and the hazard rate by CIR++ model. The description of these models can be also found in [1].

Following table contains results of this comparison without and with wrong-way risk. The hazard rate is fixed and the swap price development satisfies G2++ model. The results of the semi-analytical approach are in column $\mathrm{CVA}_{\mathrm{IRS}_1}$ and column $\mathrm{CVA}_{\mathrm{IRS}_2}$ corresponds to the results of the simulation study. Please note that the correlation used in [2] is an instantaneous correlation of the interest rate and the hazard rate denoted by $\overline{\rho}$ whilst we are using the correlation between absolute values of the interest rate and the default time.

Correlation	Hazard rate	$\mathbf{CVA_{IRS_1}}$	$\mathbf{CVA_{IRS_2}}$
$\rho = 0$, $\overline{\rho} = 0$	3 %	0.222 %	0.22 %
	5 %	0.343 %	0.34 %
	7 %	0.447 %	0.44 %
$\rho = 1$, $\overline{\rho} = -1$	3 %	0.841 %	0.36 %
	5 %	1.027 %	0.46 %
	7 %	1.074 %	0.54 %

Table 2: IRS prices including CCR with and without wrong-way risk

The results without wrong-way risk (zero correlation) are almost the same which we have expected. The results with the wrong-way risk are different which was also expected because the correlations does not express the same dependence. Although our approach provides more conservative results, it can be expected that after the correlation coefficients calibration on the same market data, both methods give similar results.

9 Conclusion

In this paper we have noted that the CVA of the OTC derivative including wrong-way risk can be expressed as a difference between the covariance and the CVA without wrong-way risk. It is a general formula without any additional assumptions on the model of the default time nor the underlying asset. But it can be simplified assuming linear dependence represented by the constant correlation coefficient.

Main part of this paper deals with the development of the semi-analytical formula for the IRS price calculation including CCR. The formula uses the constant correlation Gaussian copula dependence of the default time and interest

rate based on the [5]. Using this formula we also found that wrong-way risk has relevant impact on the OTC IRS price with lower hazard rate, resp. probability of default, which has been analyzed in the numerical study where we evaluate 10Y plain vanilla IRS.

We compared the semi-analytical formula with the simulation study presented in [2]. In case of no wrong-way risk both approaches give almost the same results. But if we include the wrong-way risk the results vary. This difference is caused by different calculation of the correlation coefficient. In [2] the correlation is instantenous between the hazard rate and interest rates changes, in this paper the correlation is between (absolute) values of the default time and the interest rate.

The correlation between absolute values gives, in our view, better information of the dependence than the instantaneous correlation. On the other hand the calibration of the instantaneous correlation is much easier because the hazard rates are observable on the market from the CDS spreads. The calibration of the correlation between the default time and the interest rate (or other underlying assets) is a subject of the further research.

Other possible disadvantages of the semi-analytical formula are constant LGD, Gaussian copula dependence and constant correlation coefficient assumptions. For more accurate calculation of the CVA the LGD should be stochastic taking into account economic cycles, especially downturn periods. LGD has typically U-shaped distribution which is mostly modeled by the beta distribution. The use of the Gaussian copula does not correspond to the situation on the market where heavy-tailed distribution should be assumed (see [3]). In general the correlation should be stochastic so the constant correlation assumption is not realistic. As a compromise we can assume the term structure of the correlation which leaves the formula almost unchanged (except the correlation).

Despite all these weaknesses of the formula it is very quick and simple method to implement and to calculate the IRS CVA with and without wrong-way risk.

Appendix

Maturity Date	Rate	Maturity Date	Rate	Maturity Date	Rate
26-Jun-06	2.83%	20-Sep-07	3.46%	27-Jun-16	4.19%
27-Jun-06	2.83%	19-Dec-07	3.52%	27-Jun-17	4.23%
28-Jun-06	2.83%	19-Mar-08	3.57%	27-Jun-18	4.27%
04-Jul-06	2.87%	19-Jun-08	3.61%	27-Jun-19	4.31%
11-Jul-06	2.87%	18-Sep-08	3.65%	29-Jun-20	4.35%
18-Jul-06	2.87%	29-Jun-09	3.75%	28-Jun-21	4.38%
27-Jul-06	2.88%	28-Jun-10	3.84%	27-Jun-22	4.41%
28-Aug-06	2.92%	27-Jun-11	3.91%	27-Jun-23	4.43%
20-Sep-06	2.96%	27-Jun-12	3.98%	27-Jun-24	4.45%
20-Dec-06	3.14%	27-Jun-13	4.03%	27-Jun-25	4.47%
20-Mar-07	3.27%	27-Jun-14	4.09%	29-Jun-26	4.48%
21-Jun-07	3.38%	29-Jun-15	4.14%	28-Jun-27	4.50%

Table 3: EUR zero-coupon continuously-compounded spot rates (ACT/360) observed on June,23 2006.

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