

Identifying Price Jumps from Daily Data with Bayesian vs. Non-Parametric methods

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Abstract

Non-parametric approach to financial time series jump estimation, using the *L*-Estimator, is compared with the parametric approach utilizing a Stochastic-Volatility-Jump-Diffusion (SVJD) model, estimated with MCMC and extended with Particle Filters to estimate the out-sample evolution of its latent state variables, such as the jump occurrences. The comparison is performed on simulated time series with different kinds of dynamics, including Poisson jumps, self-exciting Hawkes jumps with long-term clustering, as well as co-jumps. In addition to that, a comparison is performed on the real world daily time series of 4 major currency exchange rates. The results from the simulation study show that for the purposes of in-sample estimation does the MCMC based parametric approach significantly outperform the *L*-Estimator. In the case of the out-sample estimates, based on a combination of MCMC and Particle Filters, used to sequentially estimate the jump occurrences immediately at the times at which the jumps occur, does the parametric approach achieve a similar accuracy as the non-parametric one in the case of the simulations with Poisson jumps that are relatively large, and it outperforms the non-parametric approach in the case of Hawkes jumps when the jumps are large. On the other hand, the *L*-Estimator provides better results than the parametric approach in all of the cases when the simulated jumps are small (1% or less), regardless of the jump process dynamics. The application of the methods to foreign exchange rate time series further shows that the estimates of the parametric method may be biased in the case when large outlier jumps occur in the time series as well as when the stochastic volatility grows too high (as happened during the crisis). In both of these cases, the non-parametric *L*-Estimator based approach seems to provide more robust jump estimates, less influenced by the mentioned issues.

AMS/JEL classification: C11, C14, C15, C22, C58, G1

Keywords: Asset price jumps, power variation estimators, *L*-Estimator, Bayesian estimation, SVJD, MCMC, Particle filters, Hawkes process, Self-exciting jumps

Acknowledgments

This paper has been prepared under financial support of a grant “Advanced methods of financial asset returns and risks modelling” IGA VŠE F1/23/2015 which the author gratefully acknowledges.

Introduction

The identification and modelling of asset price jumps (i.e. discontinuous price changes) plays an important role in many areas of finance such as risk management, asset pricing, derivatives valuation and quantitative trading. The jumps significantly increase the size of the tails of the short-horizon return distribution, thus playing an important role in short-horizon VaR calculation (Witzany, 2013) and short-maturity option pricing (Fulop, Li and Yu, 2014). In addition to that, the jumps exhibit a different dynamics than the continuous price volatility, most notably self-excitation and size-dependency (Fičura and Witzany, 2016), playing an important role in volatility forecasting (Corsi, Pirino, Reno, 2010). Additionally, the jumps have a negative impact on certain market making strategies, they violate the continuous semi-martingale condition, negatively influencing the option delta-hedging strategies, they increase the option volatility risk premium (Todorov, 2010) and they also seem to carry a momentum-like potentially profitable trading signals (Novotny et. al., 2015).

The main problem for jump modelling and forecasting is that the jumps are unobservable and they have to be disentangled from the continuous component of the asset price variability. If the price was observable at an arbitrarily high (i.e. continuous) frequency, the identification of jumps would be straightforward, but in real-life setting, with the price available only in discrete time-points, the identification of jumps poses a challenging problem.

A commonly used approach in the case when only the daily asset return time series are available is to formulate a stochastic-volatility jump-diffusion (SVJD) model of the asset price behaviour and to estimate the jump occurrences and jump sizes as time series of latent state variables, together with the model parameters and the latent state time series of the return variances, by using Bayesian inference methods such as MCMC (Markov-Chain Monte-Carlo) and Particle Filters (PF). The MCMC based approach was used for example in Eraker, Johannes and Polson (2003) or Witzany (2013). While in previous studies jumps were usually assumed to be independent, following a Poisson process with constant intensity, in the recent years, models that extend the dynamics of the jump component have been proposed, often using the self-exciting Hawkes process in order to model the jump clustering effects (Ait-Sahalia et al. 2013, Fulop, Li and Yu 2014 or Fičura and Witzany 2016).

With the increased availability and quality of intraday financial data, further methods have been proposed for jump identification and modelling using the intraday price returns. In the parametric setting, Stroud and Johannes (2014) propose a Bayesian SVJD model working on the 5-minute frequency utilizing jumps in returns as well as volatility, while Fičura and Witzany (2015a) introduce a SVJD model working on the 4-hour frequency being able to model jump self-exciting effects as well as the intraday seasonality in jump intensity. Nevertheless, the application of Bayesian methods to intraday frequency time series poses significant difficulties due to the increased computational demands as the intraday time series often contain tens of thousands of observations, causing the Bayesian MCMC estimation to be time-demanding.

After the introduction of the realized variance estimator (Andersen and Bollerslev, 1998) and the further research in the area of asymptotic theory of power variations (Barndorff-Nielsen and Shephard, 2004) a new family of methods for jump identification emerged and quickly became highly popular. Unlike the parametric approach using SVJD models and Bayesian estimation methods to estimate their latent states, these new, power-variation based estimators, are non-parametric, model-free and computationally very simple, while at the same time being highly accurate. In Barndorff-Nielsen and Shephard (2004), the so called Z-Estimator was proposed, identifying if jumps occurred during a give time-period in the time-series by using the normalized difference between the realized variance and bipower variation, with the normalization performed with an estimate of the

integrated quarticity of the price process. While the realized variance represents an estimate of the total quadratic variation of the price process, measuring the continuous as well as the discontinuous price variability, the bipower variation converges only to the integrated variance, corresponding to the continuous price variability. The normalized difference between the realized variance and the bipower variation can thus be used to estimate whether jumps occurred in the given time period.

Lee and Mykland (2008) further proposed an estimator based on the local volatility of the price process, estimated with the bipower variation, which unlike the Z-Estimator is able to identify price jumps at the exact time at which they occur. Additionally, while the estimator was developed primarily for the intraday frequency time-series, it can successfully be applied to identify jumps even from the daily-frequency time series, in the cases when the higher frequencies are not available.

In this study, the parametric Bayesian approach of jump identification is compared with the non-parametric approach utilizing the L-Estimator, both applied to the daily frequency time series. The comparison is performed on simulated time series as well as on empirical time series of daily foreign exchange rate returns. The daily frequency was chosen primarily because the high-frequency intraday data may not be available in all cases and in practical applications it would then be useful to know whether it pays off, to use the computationally demanding Bayesian methods of jump estimation, or if it sufficiently accurate to utilize the easy-to-implement *L*-Estimator. The second reason why the daily frequency was chosen is because the jumps on the intraday frequencies exhibit different properties than the ones visible on the daily frequency, as was shown for example in Ficura and Witzany (2015b). Most importantly, there is a very high number of jumps on the intraday frequencies with most of them being rather small. Although such small jumps may be of some interest to the intraday traders and market makers, they are of virtually no interest to the investors working with longer time horizons as they do not influence significantly the daily return distribution. Therefore they can be ignored in certain applications, with focus being placed solely on the large jumps that are influencing the daily returns.

The rest of the paper is organized as follows. In the second chapter the non-parametric approach of jump estimation based on the L-Estimator is explained. In the third chapter the Bayesian approach based on the SVJD model is described, as well as its MCMC and SIR Particle Filter estimation methods. In the fourth chapter the simulation study is realized and the results of the parametric and the non-parametric methods are compared, while in the fifth chapter the comparison on real empirical foreign exchange time series is performed. In the last chapter conclusions are made and further areas of research are proposed.

1. Non-Parametric approach to jump identification

In this section we explain the non-parametric approach of jump identification proposed by Lee and Mykland (2008). Although the method was developed for intraday time series, it can be applied to daily frequency as well and the authors even give recommended value of the k parameter used for local volatility calculation (with bipower variation) for the daily frequency (16 days).

In order to explain the logic of the L-Estimator and bipower variationm we will further assume the following generally defined stochastic-volatility jump-diffusion process of logarithmic asset price evolution:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + j(t)dq(t) \quad \text{Eq. 1}$$

Where $p(t)$ is the logarithm of the asset price, $\mu(t)$ is the instantaneous drift rate, $\sigma(t)$ is the instantaneous volatility, $W(t)$ is a Wiener process, $j(t)$ is a process determining the jump sizes and

$q(t)$ is a counting process determining the jump occurrences. The proposed approach is model-free, which means that it is valid for a wide range of possible processes $\mu(t)$, $\sigma(t)$, $j(t)$ and $q(t)$.

The total variability of the process governing the asset price behaviour over a period of time between $t - k$ and t can be expressed with its *quadratic variation* as follows:

$$QC(t - k, t) = \int_{t-k}^t \sigma^2(s)ds + \sum_{t-k \leq s < t} \kappa^2(s) \quad \text{Eq. 2}$$

Where $QC(t - k, t)$ denotes the quadratic variation for the period between $t - k$ and t and $\kappa(s) = j(t)I[q(t) = 1]$ with $I(\cdot)$ being an indicator function. The first term on the right side, $\int_{t-k}^t \sigma^2(s)ds$, represents the continuous component of the asset price variability and is commonly called *integrated variance*, while the second term, $\sum_{t-k \leq s < t} \kappa^2(s)$, measures the overall impact jumps during the given time period, corresponding to the discontinuous price variability, and is commonly called *jump variance*. It is thus possible to rewrite the equation as:

$$QC(t - k, t) = IV(t - k, t) + JV(t - k, t) \quad \text{Eq. 3}$$

Where $IV(t - k, t)$ is the integrated variance and $JV(t - k, t)$ is the jump variance.

The non-parametric power-variation based estimators of volatility and jumps, that we utilize in this study, use the high-frequency returns (or daily returns, if high-frequency returns are not available), and the asymptotic theory of power variations, to derive estimators of $QC(t - k, t)$ and $IV(t - k, t)$, and by using their values then estimate also the $JV(t - k, t)$, measuring the effect of jumps.

The most commonly used estimator of the quadratic variation is the realized variance, defined as the sum of squared high-frequency returns over a certain period, which does asymptotically converge, with increasing sampling frequency of the returns, to the quadratic variation of the price process over the given period.

To estimate the integrated variance one can then use the so called bipower variation, which is a sum of the multiples of all of the pairs of subsequent absolute returns over a certain period. As the possible jumps are, in this case, multiplied with a continuous return whose absolute magnitude converges with increasing sampling frequency towards zero, the realized bipower variation converges to the integrated variance of the price process.

The popular Z-statistic jump estimator, proposed by Barndorff-Nielsen and Shephard (2004) is able to identify whether jumps occurred in a given time interval, by appropriately normalizing the relative differences between the realized volatility and the integrated variance with the normalization performed by using an estimate of the integrated quarticity, i.e. $\int_{t-k}^t \sigma^4(s)ds$ (usually constructed by realized tripower quarticity), corresponding to the variability of the realized variance estimate. The Z-estimator has, in the absence of jumps, standard normal distribution, so that too large values of the estimator (i.e. larger than an appropriate quantile of the standard normal distribution) indicate that a jump (or multiple jumps) occurred during the given time period.

As the goal of our current study is to identify jump at the exact time at which they occur (and not just tell if any jumps occurred in a given time period) the aforementioned Z-Estimator is not suitable for our purposes and would need to be modified in order to be used (see Hanousek et al. 2011).

Therefore we utilize the so called L-Estimator, proposed by Lee and Mykland (2008), which is based on asset price returns, normalized with the local variance estimated with the bi-power variation, in order to identify jumps in the financial time series at the exact time at which they occur.

To introduce the L -Estimator it is first necessary to define the local variance, which will in our case be estimated with Bipower Variation calculated over the last k periods (in our case days). Bipower variation over the period between $i - K$ and $i - 1$ can be defined as follows:

$$\sigma_{BV}^2(i) = BV(i - k, i - 1) = \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} |r(j)| |r(j-1)| \quad \text{Eq. 4}$$

Where $\sigma_{BV}^2(i)$ is the local variance estimated for the day i , $r(j)$ are the past logarithmic returns, defined as $r(j) = p(j) - p(j-1)$ with $p(j)$ being the logarithm of the price at day j , and k is the period (i.e. number of days) used for local variance estimation. $BV(i - k, i - 1)$ is the bipower variation over the period between $i - k$ and $i - 1$, which converges (with increasing sampling frequency of the returns) towards the integrated variance $IV(i - k, i - 1)$.

Local volatility $\sigma_{BV}(i)$ can then be used to define the L -Estimator as follows:

$$L(i) = \frac{r(i)}{\sigma_{BV}(i)} \quad \text{Eq. 5}$$

Where $L(i)$ is the L -Statistics, $r(i)$ is the return in the given time-period (i.e. day) and $\sigma_{BV}(i)$ is the local volatility (local standard deviation which is the square root of the local variance $\sigma_{BV}^2(i)$ estimated with the bipower variation).

The method of jump identification based on the L -Statistic does further utilize the known distribution of the appropriately normalized maximum value of the L -Statistics in a time series A_n of a length n , derived under the assumption that the series does not contain any jumps. The normalized maximum value of $L(i)$ can be expressed as:

$$\xi = \frac{\max_{i \in A_n} |L(i)| - C_n}{S_n} \quad \text{Eq. 6}$$

With A_n being a set of all of the time-periods in the time series, i.e. $i \in \{1, 2, \dots, n\}$, which do not contain jumps. The constants C_n and S_n are defined as follows:

$$C_n = \frac{[2 \log(n)]^{1/2}}{c} - \frac{\log(\pi) + \log[\log(n)]}{2c[2 \log(n)]^{1/2}} \quad \text{Eq. 7}$$

$$S_n = \frac{1}{c[2 \log(n)]^{1/2}} \quad \text{Eq. 8}$$

And c is a constant equal to $c = \sqrt{2}/\sqrt{\pi} \approx 0.7979$.

To identify jumps in the time series it is necessary to utilize the known distribution of the maximum ξ in a time series of length n that does not contain any jumps. The distribution of ξ in this case is:

$$P(\xi \leq x) = \exp(-e^{-x}) \quad \text{Eq. 9}$$

Jumps can thus be identified in the analysed time series as the normalized values of $L(i)$, with the normalization performed according to Eq. 6, that are larger than some sufficiently high quantile of ξ .

The jump estimation approach using the L -statistics has two meta-parameters, first is the period k , used for local volatility estimation and second is the quantile α used as a confidence level for the jump detection. In our case $k = 16$ is used in the empirical study, as recommended by Lee and Mykland (2008) for daily-frequency time series. Alternative values of k will also be tested in the simulation part of the study in section 4, in order to examine the impact of this parameter on the accuracy of the jump estimation method for time series with different patterns of jump dynamics.

As a confidence level for jump identification $\alpha = 0.90$ is used in the cases when exact times of jump occurrences are needed. In most parts of the study, however, we will work with the inferred jump probabilities of occurrence instead of the finally identified jumps.

2. Bayesian method of jump identification

The Bayesian method of jump identification used in this study is fully parametric. A Stochastic-Volatility Jump-Diffusion (SVJD) model will be specified with the times of jump occurrences being subsequently identified with an MCMC estimation as a series of latent state variables. The SVJD model is modelling stochastic volatility together with the dynamics of the jumps, which makes the method comparable with the non-parametric jump estimation approach, based on the *L*-Estimator, which is also working with stochastic volatility in the time series, therefore normalizing the asset price returns with local volatility before the jumps are estimated.

In addition to the stochastic volatility, we utilize a self-exciting Hawkes process for the modelling of the times of jump occurrences and the jump intensity, in order to account for the jump clustering effects as well as the co-jump effect (occurrence of jumps in two subsequent time periods) in the analysed time series. The Hawkes process is a popular alternative to the traditionally used Poisson process in SVJD model, which assumes the jump occurrence times to be independent and is therefore not able to account for the jump clustering and the co-jumps.

The proposed SVJD model contains 3 equations, one for the process governing the behaviour of the logarithmic price, one for behaviour of the stochastic volatility (modelled with the log-variance model utilizing an Ornstein-Uhlenbeck process for the logarithm of the stochastic variance) and one for the jump intensity (utilizing the self-exciting Hawkes process with exponential decay function).

The parameters of the model as well as the 3 latent state time series in the model (stochastic variances, jump occurrences and jump sizes, as the jump intensities which are also latent, depend deterministically on the jump sizes) will be estimated with a Markov-Chain Monte-Carlo (MCMC) algorithm in the in-sample period. To construct out-sample predictions of the jump occurrences, immediately at the time of their occurrence, a Sequential Importance Resampling (SIR) particle filter will be used to sequentially estimate the distribution of the latent state variables on every day of the out-sample period, utilizing the parameters estimated with MCMC in the in-sample period.

In continuous time the model is defined as follows.

$$dp(t) = \mu dt + \sigma(t)dW(t) + j(t)dq(t) \quad (1)$$

Where $p(t)$ is the logarithm of the asset price, μ is the constant return mean, $\sigma(t)$ is the instantaneous stochastic volatility, $W(t)$ is a Wiener process, $j(t)$ is a process determining the jump sizes and $q(t)$ is a counting process determining the times of jump occurrences.

For the modelling of the stochastic volatility $\sigma(t)$, log-variance model will be used in which the logarithm of the stochastic variance, denoted as $h(t) = \ln[\sigma^2(t)]$, follows a mean-reverting Ornstein-Uhlenbeck process expressed as follows:

$$dh(t) = \kappa[\theta - h(t)]dt + \xi dW_V(t) \quad (2)$$

Where $h(t) = \ln[\sigma^2(t)]$ is the logarithm of the stochastic variance, κ is a parameter determining the speed of the mean-reversion, θ is the long-term level of the stochastic log-variance, ξ is the volatility of the stochastic log-variance and $W_V(t)$ is a separate Wiener process governing the evolution of the

stochastic log-variance, which in our case will be uncorrelated with the Wiener process in the log-returns equation $W(t)$ as we apply the models to the foreign exchange markets in which no long-term unconditional correlation between volatility and price behaviour seems to be present (Franses and van Dijk 2000). It is apparent from the equation (16) that we also assume a constant level of the volatility of the log-variance, constant speed of mean-reversion and a constant long-term level of the log-variance toward which the process is converging. Additionally, we do not account for the possible jumps in the log-variance process which are sometimes also being modelled (see Eraker et al. 2003).

The process of jump sizes, $j(t)$, will in our case be modelled as a series of independent normally distributed random variables with constant volatility, i.e. $j(t) \sim N(\mu_J, \sigma_J)$, i.e. ignoring the possibility of a time-varying jump volatility and the possible correlation between the volatility of the jumps and the level of the stochastic variance.

The process of jump occurrences $q(t)$ will be modelled with a self-exciting Hawkes process with an exponential decay function (see Fulop, Li and Ye 2014). The jump intensity, defined as $\Pr[dq(t) = 1] = \lambda(t)dt$, is then governed by the following process:

$$d\lambda(t) = \kappa_J[\theta_J - \lambda(t)]dt + \xi_Jdq(t) \quad (3)$$

Where $\lambda(t)$ is the level of jump intensity defined as $\Pr[dq(t) = 1] = \lambda(t)dt$ (corresponding to the probability of a jump occurrence at the given time), θ_J denotes the long-term level of jump intensity, ξ_J represents an increase of jump intensity following a jump occurrence and κ_J is the speed of the exponential decay of the elevated jump intensity towards its long-term level. The model is thus able to model the self-exciting behaviour of the jumps in which the occurrence of a jump increases the future jump intensity which then decays exponentially back towards its long-term level. This enables the model to capture the effects of jump clustering, associated with κ_J closed to one and relatively low ξ_J , corresponding to the existence of stressed periods with temporarily increased jump intensity, as well as the effects of co-jumps, characteristic by a relatively high value of the self-excitation parameter ξ_J and a relatively low value of κ_J , causing the jumps to significantly increase the probability of a future jump occurrence in the immediate period after the initial jump occurrence, but not so much in the periods afterwards.

By solving the Hawkes process differential equation it is further possible to express the level of jump intensity at any time point t :

$$\lambda(t) = \theta_J + \int_{-\infty}^t \xi_J e^{-\kappa_J(t-s)} dq(s) = \theta_J + \sum_{dq(s)=1, s \leq t} \xi_J e^{-\kappa_J(t-s)} \quad (4)$$

Although the model is formulated for the continuous time, in order to estimate its parameters and use it to make predictions it is necessary to discretize it. The discretization is performed by using the Euler method, replacing the differentials in equations 15, 16 and 17 by the daily differences and rewriting the parameters. The discretized version of the model is presented below.

The discrete version of the log-return equation (15) can be expressed as follows:

$$r(t) = \mu + \sigma(t)\varepsilon(t) + J(t)Q(t) \quad (5)$$

With $r(t)$ being the daily return defined as $r(t) = p(t) - p(t-1)$ with $p(t)$ being the logarithm of the asset price at day t . Parameter μ represents the unconditional mean of the daily logarithmic

returns, $\sigma(t)$ is the conditional volatility at day t , $\varepsilon(t) \sim N(0,1)$ is a standard normal random variable (*i.i.d.* white noise), $J(t) \sim N(\mu_J, \sigma_J)$ is a series of normally distributed random variables with parameters μ_J and σ_J (determining the distribution of jump sizes), and $Q(t) \sim \text{Bern}[\lambda(t)]$ is a Bernoulli distributed random variable with intensity (probability of jump occurrence) $\lambda(t)$ at time t .

The discrete version of the Ornstein-Uhlenbeck process in equation (16), governing the evolution of the conditional log-variance, corresponds an AR(1) process (after discretization):

$$h(t) = \alpha + \beta h(t-1) + \gamma \varepsilon_V(t) \quad (6)$$

Where $h(t) = \ln[\sigma^2(t)]$ is the logarithm of the conditional variance at day t , α is the intercept of the AR(1) process, with long-term log-variance θ defined as via $\theta = \alpha/(1 - \beta)$, parameter β is the autoregressive coefficient of the model (slope of the regression), γ is the volatility of the log-variance and $\varepsilon_V(t) \sim N(0,1)$ is an *i.i.d.* standard normal white noise variable uncorrelated with $\varepsilon(t)$.

Finally, the discretized version of the Hawkes process, determining the daily jump intensity $\lambda(t)$, can be expressed as follows:

$$\lambda(t) = \alpha_J + \beta_J \lambda(t-1) + \gamma_J Q(t-1) \quad (7)$$

Where $\lambda(t)$ denotes the jump intensity at day t , $\alpha_J = (1 - \beta_J - \gamma_J)\theta_J$ determines the long-term equilibrium jump intensity θ_J via $\theta_J = \alpha_J/(1 - \beta_J - \gamma_J)$, parameter β_J determines the rate of the exponential decay of the jump intensity towards its long-term level θ_J , and γ_J determines the increase of jump intensity in the day following a jump occurrence. In the case of a process exhibiting co-jump behaviour but no long-term jump clustering, γ_J would be significantly positive and β_J would be close to zero, while in a case of a process with long-term jump clustering β_J would be close to one and γ_J relatively low, but still significantly higher than zero.

The final model does thus contain 3 equations governing the asset price dynamics (equations 19, 20 and 21), 9 model parameters to be estimated ($\mu, \alpha, \beta, \gamma, \theta_J, \beta_J, \gamma_J, \mu_J, \sigma_J$) and 3 series of latent state variables, \mathbf{V} , \mathbf{J} , and \mathbf{Q} , representing the daily conditional variance $V(t)$, defined as $V(t) = \sigma^2(t)$, series of jump sizes $J(t) \sim N(\mu_J, \sigma_J)$ and jump occurrences $Q(t) \sim \text{Bern}[\lambda(t)]$.

In the in-sample period, the latent state variables will be estimated together with the model parameters with a MCMC (Markov-Chain Monte-Carlo) algorithm, inspired by algorithm in Witzany (2013), based on earlier results in Jacquier et al. (2007) and Johannes and Polson (2009).

MCMC is a Bayesian estimation method that enables us to sample from the high-dimensional joint posterior density of the model parameters and the latent states, denoted as $p(\Theta|\text{data})$, with $\Theta = (\theta_1, \dots, \theta_k)$ being the vector of all of the model parameters and latent states, by constructing a Markov Chain that converges to this joint posterior density, while using only the information about the low-dimensional conditional densities $p(\theta_j|\theta_i, i \neq j, \text{data})$ (i.e. the conditional densities of one specific parameter or latent state, conditional on all of the other parameters, latent states and the data), that are far easier to analytically express and sample from.

MCMC is a name of a whole family of algorithms. The most straightforward one is the *Gibbs sampler*, that can be used to sample from the joint posterior density $p(\Theta|\text{data})$ in the case when we are able to sample directly from the conditional densities $p(\theta_j|\theta_i, i \neq j, \text{data})$, which may not always be the case. The Gibbs sampler proceeds as follows:

0. Assign a vector of initial values to $\Theta^0 = (\theta_1^0, \dots, \theta_k^0)$ and set $j = 0$

1. Set $j = j + 1$
2. Sample $\theta_1^j \sim p(\theta_1 | \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})$
3. Sample $\theta_2^j \sim p(\theta_2 | \theta_1^j, \theta_3^{j-1}, \dots, \theta_k^{j-1}, \text{data})$
- ...
4. Sample $\theta_k^j \sim p(\theta_k | \theta_1^j, \theta_2^j, \dots, \theta_{k-1}^j, \text{data})$ and return to step 1.

According to the Clifford-Hammersley theorem (Johannes and Polson 2009), the univariate conditional densities $p(\theta_j | \theta_i, i \neq j, \text{data})$ fully characterize the joint density $p(\Theta | \text{data})$ and it can be proved that the Markov Chain constructed according to the Gibbs Sampler converges to the joint density $p(\Theta | \text{data})$. The distribution of the parameters and the latent states of our model can thus be estimated by calculating enough iteration of the Gibbs Sampler, discarding the ones at the beginning of the sample, where the algorithm did not converge yet, and use the remaining ones as samples from the target distribution in order to estimate all of the necessary quantities that we want.

Typically, either the sample mean or the sample median are used to estimate the parameters of the model and the values of the latent state variables. Sample standard deviations can then be used to estimate the Bayesian standard errors of the parameter estimates that can be used to evaluate their statistical significance

The conditional densities $p(\theta_j | \theta_i, i \neq j, \text{data})$ necessary for the Gibbs sampler are typically derived by applying the Bayes theorem to the likelihood function and the prior density. More generally, the following proportionale relationship can be defined:

$$p(\theta_1 | \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data}) \propto L(\text{data} | \theta_1, \theta_2^{j-1}, \dots, \theta_k^{j-1}) * \text{prior}(\theta_1 | \theta_2^{j-1}, \dots, \theta_k^{j-1}) \quad (8)$$

With $L(\cdot)$ denoting the likelihood function, $\text{prior}(\cdot)$ denoting the Bayesian prior density of the given parameter and \propto corresponding to the proportionale relationship. If no prior information is available, the uninformative prior densities, $\text{prior}(\theta_i) \propto 1$ can be used for the prior. Additionally the independence of the model parameters is often assumed.

In order to derive the conditional density $p(\theta_j | \theta_i, i \neq j, \text{data})$ for the use in the Gibbs Sampler, it is necessary to further normalize the right hand side of the equation (22), by dividing it with its integral over θ_1 , corresponding to the density $p(\text{data} | \theta_2^{j-1}, \dots, \theta_k^{j-1})$, thus replacing the proportionale relationship with equality.

Nevertheless, the integration of the right hand side of (22) over θ_1 may often not be feasible in practice. In such cases the Gibbs sample cannot be used (in its standard form) and an alternative method, such as the *Metropolis-Hastings algorithm*, has to be used instead.

Metropolis-Hastings algorithm is a rejection sampling algorithm, sampling a proposal value of the given parameter from a proposal density at first, and then either accepting or rejecting it, based on a given probability.

Specifically, to utilize Metropolis-Hastings algorithm the Step 2 in the Gibbs Sample algorithm described above has to be replaced by the following two step procedure:

- A. Sample θ_1^j from a proposal density $q(\theta_1 | \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})$
- B. Accept θ_1^j with probability $\alpha = \min(R, 1)$, with R denoting the so called *acceptance ratio* defined as:

$$R = \frac{p(\theta_1^j | \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data}) q(\theta_1^{j-1} | \theta_1^j, \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})}{p(\theta_1^{j-1} | \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data}) q(\theta_1^j | \theta_1^{j-1}, \theta_2^{j-1}, \dots, \theta_k^{j-1}, \text{data})} \quad (9)$$

Which may in practice be evaluated by sampling $u \sim U(0,1)$ from an uniform distribution and accepting the value of θ_1^j if and only if $u < R$ (otherwise the value of the parameter from the previous iteration θ_1^{j-1} is kept instead).

Similarly as in the case of the Gibbs Sampler, it can be shown that the so constructed Markov-chain converges to the joint density $p(\Theta | \text{data})$ (Johannesson and Polson, 2009).

The computational efficiency of the Metropolis-Hastings algorithm does, however, severely depend on the choice of the proposal density. A wide variety of different versions of the algorithm were thus created, differing in the way in which the proposal density is defined.

A simple, easy to implement version of the algorithm is the so called *Random-Walk Metropolis-Hastings*, in which the proposal density follows a Random Walk through the parameter space. The proposal is thus defined as follows:

$$\theta_1^j \sim \theta_1^{j-1} + N(0, c) \quad (10)$$

With c being the step-size meta-parameter which may influence the computatational efficiency of the algorithm significantly and the practice is to set it so that approximately 50% of the proposals get accepted and 50% rejected.

A significant advantage of the Random-Walk Metropolis-Hastings algorithm is its feature that the proposal distribution is symmetric, in the sense that the probability of going from θ_1^{j-1} to θ_1^j is the same as the probability of going from θ_1^j to θ_1^{j-1} (which is not generally the case for most other kinds of proposal densities). This causes the proposal densities q in the acceptance ratio, in equation 23, to cancel out. Consequently, by utilizing the relationship 22 and assuming non-informative priors, the acceptance ratio reduces to likelihood ratio:

$$R = \frac{L(\text{data} | \theta_1^j, \theta_2^{j-1}, \dots, \theta_k^{j-1})}{L(\text{data} | \theta_1^{j-1}, \theta_2^{j-1}, \dots, \theta_k^{j-1})} \quad (11)$$

Therefore, as long as we are able to express the likelihood function of the model, it is possible to utilize the Random-Walk Metropolis-Hasting algorithm to estimate the joint posterior density of the model parameters and latent states.

The most difficult and time-consuming part of the SVJD model estimation is the sampling of the latent states of the stochastic variances. In our application this is performed with a specific version of the *Accept-Reject Gibbs Sampler*, proposed by Kim, Shephard and Chib (1998). As the conditional distribution of the stochastic variances $p(V_i | V_{(-i)}, \Theta, \mathbf{r}, \mathbf{J}, \mathbf{Z})$ cannot be analytically expressed and sampled from, the authors propose to use a proposal distribution q , whose density is at all points above the target density p . The Accept-Reject Gibbs Sampler does then sample in each step repeatedly from q , with the proposal being accepted with an acceptance ratio equal to the ratio of the two densities at the point of the proposal. This has the effect of effectively sampling from the target density p , with the drawback that in every step potentially multiple proposals have to be sampled from q , until one of them eventually gets accepted.

As for the overall proceeding of the algorithm, we are estimating a few model parameters Θ and a large number of latent state variables X . Since we know from the Bayes theorem that:

$$p(\Theta, X | \text{data}) \propto p(\text{data} | \Theta, X) * p(X, \Theta) \quad (12)$$

We can estimate iteratively the parameters and the latent states as follows:

$$\begin{aligned} p(\Theta | X, \text{data}) &\propto p(\text{data} | \Theta, X) * p(X | \Theta) * p(\Theta) \\ p(X | \Theta, \text{data}) &\propto p(\text{data} | \Theta, X) * p(\Theta | X) * p(X) \end{aligned} \quad (13)$$

With the algorithm combining different versions of the MCMC algorithm for different variables. Specifically, the Gibbs Sampler is used for the parameters $\mu, \mu_j, \sigma_j, \alpha, \beta, \gamma$ and the latent state variables \mathbf{Q} and \mathbf{J} , the Accept-Reject Gibbs Sampler is used for the estimation of the stochastic variances \mathbf{V} and the Random-Walk Metropolis Hastings is used to estimate the parameters of the Hawkes process $\theta_j, \beta_j, \gamma_j$.

The full estimation algorithm proceeds as follows:

1. Sample reasonable initial values $\mu^{(0)}, \mu_J^{(0)}, \sigma_J^{(0)}, \alpha^{(0)}, \beta^{(0)}, \gamma^{(0)}, \theta_J^{(0)}, \beta_J^{(0)}, \gamma_J^{(0)}, \mathbf{V}^{(0)}, \mathbf{J}^{(0)}, \mathbf{Q}^{(0)}$.

Denoting s^2 as the estimate of unconditional variance $\text{Var}(r)$, the following initial values were used: $\mu^{(0)} = 0, \mu_J^{(0)} = 0, \sigma_J^{(0)} = 2 * s, \alpha^{(0)} = \log(s^2) * (1 - 0.9), \beta^{(0)} = 0.9, \gamma^{(0)} = 0.3, \theta_J^{(0)} = 0.05, \beta_J^{(0)} = 0.8, \gamma_J^{(0)} = 0.01$. The initial stochastic variances $\mathbf{V}^{(0)}$ were set equal to the exponential moving average of r^2 and jump occurrences $\mathbf{Q}^{(0)}$ were set equal to zero.

2. For $i = 1, \dots, T$ sample jump sizes $J_i^{(g)} \propto \varphi(J; \mu_J^{(g-1)}, \sigma_J^{(g-1)})$ if $Q_i^{(g-1)} = 0$ and

$$J_i^{(g)} \propto \varphi\left(r_i; \mu^{(g-1)} + J, \sqrt{V_i^{(g-1)}}\right) \varphi\left(J; \mu_J^{(g-1)}, \sigma_J^{(g-1)}\right) \text{ if } Q_i^{(g-1)} = 1$$

3. For $i = 1, \dots, T$ sample jump occurrences $Q_i^{(g)} \in \{0, 1\}, \Pr[Q = 1] = p_1/(p_0 + p_1)$, where:

$$p_0 = \varphi\left(r_i; \mu^{(g-1)}, \sqrt{V_i^{(g-1)}}\right)(1 - \lambda^{(g-1)}) \text{ and } p_1 = \varphi\left(r_i; \mu^{(g-1)} + J, \sqrt{V_i^{(g-1)}}\right)\lambda^{(g-1)}$$

4. Sample new stochastic log-variances $h_i^{(g)} = \log(V_i^{(g)})$ for $i = 1, \dots, T$ using Gibbs Sampler with accept-reject procedure developed in Kim, Shephard and Chib (1998), i.e. we calculate series $y_i = r_i - \mu^{(g-1)} - J_i^{(g)} Q_i^{(g)}$ and sample $h_i^{(g)}$ from a proposal distribution $\varphi(h_i; \mu_i, \sigma)$, where: $\mu_i = \phi_i + \frac{\sigma^2}{2} [y_i^2 \exp(-\phi_i) - 1], \phi_i = \frac{[\alpha(1-\beta)+\beta(\log V_{i+1}+\log V_{i-1})]}{(1+\beta^2)}$ and $\sigma = \frac{\gamma}{\sqrt{1+\beta^2}}$.

The proposal is accepted with probability f^*/g^* (otherwise a new proposal is drawn), where

$$\log f^* = -\frac{h_i}{2} - \frac{y_i^2}{2} [\exp(-h_i)] \text{ and } \log g^* = -\frac{h_i}{2} - \frac{y_i^2}{2} [\exp(-\phi_i)(1 + \phi_i) - h_i \exp(-\phi_i)].$$

5. Sample new stochastic volatility autoregression coefficients $\alpha^{(g)}, \beta^{(g)}, \gamma^{(g)}$ from $h_i = \log(V_i^{(g)})$ for $i = 1, \dots, T$ using the Bayesian linear regression model (Lynch, 2007), i.e.:

$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}\mathbf{y}$, $\hat{\mathbf{e}} = \mathbf{y} - \mathbf{X}\hat{\beta}$, where $\mathbf{X} = \begin{pmatrix} 1 & \cdots & 1 \\ h_1 & \dots & h_{T-1} \end{pmatrix}'$ and $\mathbf{y} = (h_2 \dots h_T)'$, so we sample:

$$(\gamma^{(g)})^2 \propto IG\left(\frac{n-2}{2}, \frac{\hat{\mathbf{e}}'\hat{\mathbf{e}}}{2}\right) \text{ and } (\alpha^{(g)}, \beta^{(g)})' \propto \varphi\left[(\alpha, \beta)'; \hat{\beta}, (\gamma^{(g)})^2 (\mathbf{X}'\mathbf{X})^{-1}\right]$$

6. Sample $\mu^{(g)}$ based on the normally distributed time series $r_i - J_i^{(g)} Q_i^{(g)}$ with variances $V_i^{(g)}$:

$$p(\mu^{(g)} | \mathbf{r}, \mathbf{J}^{(g)}, \mathbf{Q}^{(g)}, \mathbf{V}^{(g)}) \propto \varphi\left(\mu; \frac{\sum_{i=1}^T r_i - J_i^{(g)} Q_i^{(g)}}{\sum_{i=1}^T V_i^{(g)}} / \sqrt{\sum_{i=1}^T V_i^{(g)}}, \frac{1}{\sum_{i=1}^T V_i^{(g)}}\right)$$

7. Sample $\theta_J, \beta_J, \gamma_J$ using Random-Walk Metropolis-Hastings and the likelihood function:

$$L(\mathbf{Q}^{(g)} | \theta_J, \beta_J, \gamma_J) = \prod_{i=1}^T \lambda_i^{Q_i} (1 - \lambda_i)^{1 - Q_i}$$

8. Sample $\mu_J^{(g)}, \sigma_J^{(g)}$ based on the normally distributed series $\mathbf{J}^{(g)}$ and uninformative priors

$p(\mu) \propto 1$ and $p(\log \sigma^2) \propto 1$ (which is equivalent to $p(\sigma^2) \propto 1/\sigma^2$), i.e. we sample from:

$$p(\mu_J^{(g)} | \mathbf{J}^{(g)}, \sigma_J^{(g-1)}) \propto \varphi\left(\mu_J^{(g)}; \frac{\sum_{i=1}^T J_i^{(g)}}{T}, \frac{\sigma_J^{(g-1)}}{\sqrt{T}}\right)$$

$$p\left[\left(\sigma_J^{(g)}\right)^2 | \mathbf{J}^{(g)}, \mu_J^{(g)}\right] \propto IG\left[\left(\sigma_J^{(g)}\right)^2; \frac{T}{2}, \frac{\sum_{i=1}^T (J_i^{(g)} - \mu_J^{(g)})^2}{2}\right]$$

The MCMC method enables us to estimate the distribution of the parameters of the SVJD model and the evolution of the latent state variables conditional on the whole data sample. While this may be useful in many applications, it is impractical to use the method to perform out-sample forecasts, needed for example in the case of volatility forecasting or VaR estimation, as it would require to re-estimate the algorithm for every time point in the time series (using all of the data up to that time point) after which the forecasts could be generated by using simulations of the latent state evolution. As the re-estimation of the algorithm for each time point in the time series would be prohibitively time consuming in most applications, a common approach is to combine the MCMC algorithm with a Particle Filter, using the MCMC to estimate the model parameters and the evolution of the latent states in the in-sample period, while the Particle Filter can then be used to sequentially estimate the evolution of the latent states in the out-sample period (using the parameter values estimated with the MCMC).

Particle Filters (also known as Sequential Monte-Carlo) are a set of Bayesian estimation methods, that are using a weighted set of particles, together with Bayesian recursion equations, in order to sequentially estimate the posterior densities of a set of latent state variables of a given statistical model for each time point in the analysed time series.

In contrast to the MCMC method, the Particle Filters estimate the posterior density of the latent state variable at every time point conditional only on the information available up to that time point, while the MCMC method calculates posterior density estimates conditional on the whole evolution of the time series used in the estimation.

Specifically, assuming that we have an observable series y_t of a length T , governed by a set of model parameters Θ and a latent state time series x_t , the MCMC algorithm will for each time point t estimate the posterior distribution of the corresponding latent state $p(x_t | \mathcal{F}_T)$, with \mathcal{F}_T denoting the filtration (i.e. available information about the evolution of y_t) up until the end of the dataset T . The Particle Filter algorithm, on the other hand, will estimate for each time point t the posterior distribution $p(x_t | \mathcal{F}_t)$, which is conditional only on the observable information up until the time t .

The distribution $p(x_t | \mathcal{F}_t)$, defined by a weighted set of particles, can then be used, together with the model parameters, to estimate the distributions $p(x_{t+1} | \mathcal{F}_t)$, $p(x_{t+2} | \mathcal{F}_t)$, etc. by simulating the future latent state evolution. The algorithm can thus be used to sequentially estimate the posterior density of the latent states at every time point in the time series (conditional on the available information up to that time point), while simulations can then be used to generate forecasts of the future evolution of the latent states since the given time point. The algorithm does thus enable to efficiently use a SVJD model for tasks such as volatility forecasting, VaR estimation and option pricing, without the need to re-estimate the whole MCMC at every time point in the time series.

For the purposes of this study, we will use the Sequential Importance Resampling (SIR) Particle Filter, first proposed by , in order to estimate the probabilities of jump occurrence for each time-point in the analysed time series, conditional on the information available up to that time point. So constructed forecasts would then be directly comparable with the non-parametric forecasts provided by the L -Estimator, which has also the property of being able to identify jumps immediately at the time at which they occur in the time series.

To define the SIR Particle Filter, we denote y_t as observations of the observable time series, while x_t would denote observations of the unobservable, latent time series, that is governing the dynamics y_t . The purpose of the SIR Particle Filter is to estimate the posterior distribution of x_t at each time point of the time series, conditional on the evolution of x_t up to that time point as well as the known dynamics of both of the series, governed by a set of known parameters Θ .

To estimate the posterior distribution of x_t at each time point t , SIR Particle Filter uses a set of $L = 1, \dots, P$ weighted particles $x_t^{(L)}$, each of them representing one path of the possible evolution of x_t . Bayesian recursion equations are then used to adjust the weights of the particles $w_t^{(L)}$, so that they represent the posterior distribution $p(x_t | y_0, \dots, y_t)$. The values of the particles $x_t^{(L)}$ and their weights

$w_t^{(L)}$ can then be used to approximate the expectation of any desired function of x_t (such as mean, median or standard deviation), as follows:

$$\int_{-\infty}^{\infty} f(x_t) p(x_t | y_0, \dots, y_t) dx_t \approx \sum_{i=1}^P x_t^{(i)} f(x_t^{(i)}) \quad (14)$$

The SIR Particle Filter introduces a re-sampling phase into the algorithm in order to avoid the problem of degeneracy of the particles, which is a situation in which all of the weights become very close to zero except for one.

The SIR Particle Filter algorithm proceeds as follows:

1. For $L = 1, \dots, P$ particles draw samples from the proposal density $x_t^{(L)} \sim \pi(x_t | x_{0:t-1}^{(L)}, y_{0:t})$
2. For $L = 1, \dots, P$ update the importance weights up to a normalizing constant according to:

$$w_t^{*(L)} = w_{t-1}^{(L)} \frac{p(y_t | x_t^{(L)}) p(x_t^{(L)} | x_{t-1}^{(L)})}{\pi(x_t^{(L)} | x_{0:t-1}^{(L)}, y_{0:t})} \quad (15)$$

A common approach is to use proposal density equal to the conditional density of x_t , in this case $\pi(x_t^{(L)} | x_{0:t-1}^{(L)}, y_{0:t}) = p(x_t^{(L)} | x_{t-1}^{(L)})$, the weight updating equation simplifies then to:

$$w_t^{*(L)} = w_{t-1}^{(L)} p(y_t | x_t^{(L)}) \quad (16)$$

Where $p(y_t | x_t^{(L)})$ corresponds to the likelihood of y_t conditional on $x_t^{(L)}$

3. For $L = 1, \dots, P$ compute the normalized weights: $w_t^{(L)} = \frac{w_t^{*(L)}}{\sum_{J=1}^P w_t^{*(J)}}$
4. Compute effective number of particles: $N_{eff} = \frac{1}{\sum_{J=1}^P (w_t^{(J)})^2}$
5. If $N_{eff} < N_{thr}$ then resample the particles with probabilities proportional to their weights and for $L = 1, \dots, P$ set $w_t^{(L)} = 1/P$

In our study is the observable time series the evolution of the asset price returns $r(t)$, while the latent state time series are the logarithmic conditional variances $h(t) = \ln[\sigma^2(t)]$, following the log-variance model, the jump sizes $J(t)$, which are a series of normally distributed random variables, and the jump occurrences $Q(t)$, governed by the Hawkes process. The SIR particle filter sampling and weight updating equations can then be derived easily from the model dynamics equations 5, 6 and 7.

3. Comparison of the methods on simulated time series

In this section, the non-parametric approach of jump identification using the bipower variation and the L -statistics will be compared with the parametric approach utilizing a SVJD model with self-exciting jumps and Bayesian methods (specifically the MCMC to construct in-sample jump estimates and the SIR Particle Filter to construct out-sample jump forecasts), on simulated time series.

Several different tests will be performed, in order to evaluate the accuracy of the parametric and the non-parametric methods with regards to their ability to identify jumps under different dynamics of the underlying data generating process (i.e. different jump sizes as well as different types of dynamics of the self-exciting Hawkes process, etc.). All simulations will correspond to daily frequency return time series, they will utilize the presented SVJD model defined in equations 5, 6 and 7, with parameter values set in order to realistically simulate the behaviour of the foreign Exchange rate returns.

To asses the predictive power of the models, Accuracy Ratio (AR) will be utilized, enabling us to compare how well do the inferred probabilities of jump occurrence of the tested methods differentiate between jumps days and non-jump days. Accuracy Ratio (also known as Gini Coefficient) is a common measure of predictive power, used in for models with binary outcomes (such as credit scoring models, differentiating between the defaulting and no defaulting clients).

To define the Accuracy Ratio (AR), we define $p_{success}$ as the probability of successful discrimination, defined in a sense that if $Q_{i,Jump}$ denotes a random day at which a jump occurred and $Q_{j,NoJump}$ a random day at which no jump occurred, $p_{success}$ represents the probability that $E(Q_i|model) > E(Q_j|model)$, with $E(Q_i|model)$ denoting the probability of jump occurrence assigned by the model to the day $Q_{i,Jump}$ and $E(Q_j|model)$ the probability of jump occurrence assigned by the model to the day $Q_{j,NoJump}$. Similarly, we define p_{fail} as the probability, that for two randomly chosen days $Q_{i,Jump}$ and $Q_{j,NoJump}$, defined in the same fashion as above, the opposite would hold, i.e. $(Q_i|model) < E(Q_j|model)$.

Using the above probabilities $p_{success}$ and p_{fail} , it is possible to calculate the Accuracy Ratio (AR) of the model as follows:

$$AR = p_{success} - p_{fail}$$

The major advantage of the Accuracy Ratio is that it gives us a single number that can be used to compare the ability of the tested methods to discriminate between jump days and non-jump days in different settings (i.e. different simulations with different parameter values). The drawback of the statistics is, on the other hand, that it does not tell us, how well are the estimated jump intensities calibrated (i.e. if the probabilities of jump occurrences derived from the model truly correspond to the real probabilities that jump has occurred on the given day).

The simulation analysis is performed in two steps. In the first step, the predictive power of the non-parametric L-Estimator is evaluated with regards to its ability to identify jumps of different sizes in simulated time series with different dynamics, specifically regarding their self-exciting behaviour. As a part of this analysis it is evaluated, what values of the parameter k , determining the number of periods used for local volatility calculation, lead to the best results in jump identification and whether the optimal value of this parameter depends on the jump dynamics in the analysed process. In the second part of the analysis, a comparison between the non-parametric L-Estimator with the parametric SVJD based approach is performed. The parametric approach is in this case applied in its in-sample version, using MCMC estimation of the jumps, as well as in its out-sample version, using

MCMC for the in-sample parameter estimation and a SIR Particle Filter for the out-sample jump identification. The goal of the second part of the analysis is to answer the question whether the more computationally demanding parametric approach, utilizing the MCMC and Particle Filters, is able to outperform the much simpler approach of jump identification based on the L-estimator. This issue is analysed in a setting when both of the estimators are applied to a time series simulated on the daily frequency, in several variants, with varying absolute jump sizes as well as with different types of dynamics of the self-exciting Hawkes process governing the jump occurrences. Specifically, the following 3 types of dynamics will be tested: Independent Poisson jumps, Hawkes process with long-term jump clustering and a Hawkes process with short-term clustering corresponding to co-jumps.

3.1. Predictive power of the *L*-Estimator

The purpose of the simulation study in this section is to evaluate the ability of the L-Estimator to identify jumps in simulated time series with different dynamics. The focus is placed primarily on the effect of the absolute size of the jumps in the simulated time series (determined by the value of the parameter σ_{J}), on the performance of the estimator measured by the Accuracy Ratio. Additionally, the estimator is in all of the cases applied with different values of the parameter k , determining the period used for local volatility calculation, in order to see, to what degree does the choice of this parameter influence the power of the estimator to identify jumps.

When not stated otherwise, the underlying asset price process is simulated according to the SVJD model described in section 2, with parameters shown in Table 1, which more or less correspond to the typical values of parameters, observed for daily frequency foreign exchange rate time series.

Table 1 – Default parameter values used in the simulations of the asset price process

| | mui | muiJ | sigmaJ | stdLT | beta | gamma | lambdaLT | betaJ | gammaJ |
|------------------|------------|-------------|---------------|--------------|-------------|--------------|-----------------|--------------|---------------|
| Parameter | 0 | 0 | 0,01 | 0,01 | 0,99 | 0,1 | 0,05 | 0 | 0 |

In the first round of the simulations, we analyse the effect of the size of the jumps, determined by the parameter σ_{J} , and of the choice of the parameter k , on the performance of the L-Estimator, measured with the Accuracy Ratio. For this purpose, the asset price process is simulated in 10 different variants, with the value of the parameter σ_{J} ranging from 0.005 to 0.05, corresponding to average absolute jump size varying between 0.5% and 5% (compared with stochastic volatility with long-term value equal to 1% per day). The other parameters of the simulations correspond to the values shown in Table 1. As we can see, the values of β_{J} and γ_{J} are set to zero, which is the case of a Poisson process jumps with constant intensity equal to $\lambda_{\text{LT}} = 5\%$ per day.

For every simulation, the L-Estimator is applied to the simulated time series in 9 different variants, with different values of k , ranging from 4 to 60. For each simulated time series and each setting of the L-estimator, Accuracy Ratio of the jump estimates is then computed in order to evaluate how well did the given setting of the estimator identify the jumps in the given simulated time series.

To reduce the effect of the randomness of the simulations on the results of the analysis, the described procedure is repeated 200 times, with the Accuracy Ratios for each simulation and L-Estimator setting being saved. An average value of the Accuracy Ratio is then computed for every variant (i.e. every combination of σ_{J} and k). The results are shown in Table 2.

Table 2 – Average values of the Accuracy Ratios of the L -Estimator applied to 200 simulations of a SVJD process simulated in 10 variants with different values of SigmaJ

| | | SigmaJ | | | | | | | | | |
|---------|----|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| | | 0,005 | 0,01 | 0,015 | 0,02 | 0,025 | 0,03 | 0,035 | 0,04 | 0,045 | 0,05 |
| K value | 4 | 0,042 | 0,120 | 0,206 | 0,291 | 0,354 | 0,414 | 0,464 | 0,509 | 0,538 | 0,575 |
| | 6 | 0,039 | 0,124 | 0,214 | 0,304 | 0,370 | 0,435 | 0,485 | 0,531 | 0,565 | 0,599 |
| | 8 | 0,036 | 0,121 | 0,215 | 0,305 | 0,375 | 0,439 | 0,487 | 0,536 | 0,570 | 0,603 |
| | 12 | 0,037 | 0,119 | 0,211 | 0,304 | 0,376 | 0,439 | 0,488 | 0,538 | 0,570 | 0,602 |
| | 16 | 0,033 | 0,116 | 0,209 | 0,301 | 0,375 | 0,437 | 0,487 | 0,537 | 0,570 | 0,601 |
| | 24 | 0,030 | 0,112 | 0,207 | 0,297 | 0,372 | 0,435 | 0,484 | 0,535 | 0,568 | 0,599 |
| | 32 | 0,030 | 0,110 | 0,203 | 0,296 | 0,371 | 0,434 | 0,483 | 0,534 | 0,567 | 0,598 |
| | 48 | 0,029 | 0,105 | 0,199 | 0,291 | 0,367 | 0,432 | 0,479 | 0,531 | 0,563 | 0,595 |
| | 60 | 0,028 | 0,103 | 0,196 | 0,290 | 0,365 | 0,429 | 0,477 | 0,529 | 0,561 | 0,593 |

We can see from Table 2 that the accuracy ratios are increasing with the increasing average absolute size of the jumps, determined by SigmaJ. As for the values of the parameter k , the value of 16, recommended by Lee and Mykland (2008), seems to be very close to the optimal value of 12, in the case when the jumps are relatively large (i.e. larger than 1% in absolute value). In the cases when the jumps are small, however, lower values of k seem to be more suitable, reaching higher values of the Accuracy Ratios.

The results in Table 2 correspond to simulated time series where the jumps follow a Poisson process (as the betaJ and gammaJ parameters of the Hawkes process have been set to zero, corresponding to a constant jump intensity equal to lambdaLT). As already mentioned, multiple studies indicate that the jumps in the real financial time series may exhibit clustering effects, caused by the jump self-excitation (i.e. when a jump occurs, the probability of future jumps temporarily increases). In order to evaluate whether the presence of these self-exciting effects does influence the accuracy of the jump identification by the L -Estimator, two additional simulations are performed, in which the simulated jumps follow a self-exciting Hawkes process. These two simulations correspond to the two commonly observed patterns of jump self-exciting behaviour observed in the financial time series. The first pattern can be described by a relatively large value of the parameter betaJ (close to one) and a relatively low value of the parameter gammaJ, leading to relatively long-term shifts between the low and high levels of jump intensity, corresponding to calm or stressed periods on the market, associated with long-term jump clusters. The second pattern, on the other hand, can be characterized by a relatively low value of the betaJ parameter (much smaller than 1), coupled with a relatively high value of gammaJ. This parametrization causes the jump intensity to increase strongly immediately after a jump occurrence and afterwards decaying quickly to its long-term level, leading to the well known effect of co-jumps (jumps in two subsequent days) and possibly short-term high-jump-intensity clusters.

The parameter values corresponding to the first pattern, i.e. to the long-term jump clustering, are shown in Table 3.

Table 3 – Parameter values used for the simulation of the price process with self-exciting jumps and slow decay of the jump intensity (jump clustering)

| | mui | muiJ | sigmaJ | stdLT | beta | gamma | lambdaLT | betaJ | gammaJ |
|-----------|-----|------|--------|-------|------|-------|----------|-------|--------|
| Parameter | 0 | 0 | 0,01 | 0,01 | 0,99 | 0,1 | 0,05 | 0,98 | 0,015 |

The value of the parameter β_{J} was set to 0.98, so that the increased jump intensity decays only slowly (with the rate of 2% per day) to its long-term level. At the same time, the jump self-excitation parameter γ_{J} , is set to only 0.015, meaning that the occurrence of a jump increases the jump intensity on the following day by 1.5%. Table 4 shows the average Accuracy Ratios of the simulations performed by using the parameters in Table 3 (with the exception of the σ_{J} parameter, which varies between 0.005 and 0.05).

Table 4 – Average values of the Accuracy Ratios from 200 simulations of the asset price process with self-exciting jumps and slow decay of the jump intensity (jump clustering)

| | | SigmaJ | | | | | | | | | |
|------------|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| | | 0,005 | 0,01 | 0,015 | 0,02 | 0,025 | 0,03 | 0,035 | 0,04 | 0,045 | 0,05 |
| K value | 4 | 0,0362 | 0,1179 | 0,2022 | 0,2751 | 0,3427 | 0,3976 | 0,4503 | 0,4964 | 0,5254 | 0,549 |
| | 6 | 0,0372 | 0,1188 | 0,2093 | 0,2878 | 0,3548 | 0,4148 | 0,4695 | 0,5167 | 0,5455 | 0,5701 |
| | 8 | 0,0343 | 0,1183 | 0,2092 | 0,2896 | 0,3585 | 0,4181 | 0,4745 | 0,517 | 0,5485 | 0,5712 |
| | 12 | 0,0319 | 0,1142 | 0,2062 | 0,2863 | 0,3562 | 0,4164 | 0,4726 | 0,5168 | 0,5471 | 0,5703 |
| | 16 | 0,0314 | 0,1113 | 0,2044 | 0,2842 | 0,356 | 0,415 | 0,4695 | 0,5165 | 0,544 | 0,5688 |
| | 24 | 0,0305 | 0,1084 | 0,2003 | 0,2804 | 0,3517 | 0,4124 | 0,4676 | 0,5133 | 0,5412 | 0,5666 |
| | 32 | 0,0291 | 0,1058 | 0,1974 | 0,2778 | 0,3499 | 0,4102 | 0,4652 | 0,5109 | 0,5416 | 0,5647 |
| | 48 | 0,0276 | 0,1016 | 0,194 | 0,2744 | 0,3463 | 0,4084 | 0,4625 | 0,5073 | 0,5386 | 0,5622 |
| | 60 | 0,0268 | 0,1004 | 0,1919 | 0,2719 | 0,3437 | 0,4082 | 0,4603 | 0,5058 | 0,5371 | 0,5614 |

We can see from Table 4, that the results are more or less similar to the case of the Poisson jumps as shown in Table 2. Among the minor differences is that the overall size of the Accuracy Ratios slightly decreased, indicating that the self-exciting behaviour of the jumps slightly reduces the accuracy of the L -Estimator. The optimal value of the k parameter has further shifted to the value of 8, while in the case of Poisson jumps it was 12. Nevertheless, the value of 16, recommended by the authors of the L -Estimator, does still provide close-to-optimal results, with the exception of very small values of σ_{J} (0.5%), for which lower values of k seem to be more appropriate.

In the third round of the simulations, series with high-levels of self-excitation and quick decay of the elevated jump intensity to its long-term levels will be simulated, corresponding to the well-known co-jumps effect in which the jumps occur in two following time periods (in our case days). The parameters used in the simulations are shown in Table 5.

Table 5 – Parameter values used for the simulation of the price process with high levels of self-excitation of the jumps and quick decay of the jump intensity (effect of co-jumps)

| | mui | muiJ | sigmaJ | stdLT | beta | gamma | lambdaLT | betaJ | gammaJ |
|-----------|-----|------|--------|-------|------|-------|----------|-------|--------|
| Parameter | 0 | 0 | 0,01 | 0,01 | 0,99 | 0,1 | 0,05 | 0,6 | 0,1 |

The parameter values in Table 5 correspond to a situation when the occurrence of a jump leads to an increase of the jump intensity by 10% in the first day immediately following the jump occurrence, while decaying quickly (with a rate of 40% per day) in the following days. This dynamics will generate co-jumps in the simulated time series. The Accuracy Ratios achieved by the L -Estimator for the series simulated based on the parameters shown in Table 5 can be seen in Table 6.

Table 6 – Average values of the Accuracy Ratios from 200 simulations of the asset price process with highly self-exciting jumps and quick decay of the jump intensity (co-jumps)

| | | SigmaJ | | | | | | | | | |
|------------|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| | | 0,005 | 0,01 | 0,015 | 0,02 | 0,025 | 0,03 | 0,035 | 0,04 | 0,045 | 0,05 |
| K value | 4 | 0,0344 | 0,1045 | 0,1881 | 0,2591 | 0,3245 | 0,3787 | 0,4241 | 0,4553 | 0,4941 | 0,5236 |
| | 6 | 0,0348 | 0,1097 | 0,1967 | 0,2771 | 0,3453 | 0,404 | 0,4498 | 0,4827 | 0,5246 | 0,5527 |
| | 8 | 0,0328 | 0,1105 | 0,2006 | 0,2818 | 0,3522 | 0,4121 | 0,4591 | 0,4931 | 0,5353 | 0,5643 |
| | 12 | 0,0306 | 0,1092 | 0,2031 | 0,286 | 0,3571 | 0,4198 | 0,4676 | 0,5034 | 0,5454 | 0,5731 |
| | 16 | 0,0302 | 0,1086 | 0,2028 | 0,2879 | 0,3595 | 0,4246 | 0,4712 | 0,5075 | 0,551 | 0,577 |
| | 24 | 0,0286 | 0,108 | 0,2013 | 0,2875 | 0,3627 | 0,4256 | 0,4749 | 0,5099 | 0,5534 | 0,5804 |
| | 32 | 0,0274 | 0,1046 | 0,1999 | 0,2873 | 0,3629 | 0,4255 | 0,4756 | 0,5107 | 0,5539 | 0,5814 |
| | 48 | 0,0253 | 0,102 | 0,1979 | 0,2849 | 0,3616 | 0,4241 | 0,4759 | 0,5098 | 0,5531 | 0,582 |
| | 60 | 0,0249 | 0,1011 | 0,1954 | 0,2839 | 0,3595 | 0,4234 | 0,4751 | 0,5091 | 0,5534 | 0,5815 |

We can see from Table 6 that the behaviour of the Accuracy Ratios changed to a certain degree in the case of the time series containing co-jumps, when compared to the previous two simulations. Most importantly, the optimal value of the parameter k has moved in the direction of its larger values (24-60 periods), especially for the cases when the jumps are relatively large. The value of 16 proposed by the authors does, nevertheless, still provide relatively good results, with the possible exception (as in the previous two cases) of the very small absolute jump sizes (0.5%), for which the smaller values of k provide slightly better results.

3.2. Comparison of the L -Estimator with the parametric approach

In this section, we compare the non-parametric L -Estimator with the parametric approach to identify jumps in financial time series by utilizing a SVJD model estimated with Bayesian estimation methods. Specifically, a MCMC algorithm is used to estimate the parameters of the SVJD model, as well as the in-sample evolution of its latent state variables, including the jump occurrences and jump sizes. This gives us in-sample estimates of the days at which jumps in the analysed time series occurred. As it is often necessary to get jump estimates not only ex-post, over some historical period of time, but also in an out-sample fashion, at the exact times at which they occur, immediately at the time of their occurrence, an out-sample sequential approach is tested as well, utilizing a MCMC algorithm to estimate the parameters and the latent states of the SVJD model in the in-sample period, after which a SIR Particle Filter is used, to sequentially estimate the latent states of the model, including the jump occurrences, in the out-sample period. This approach can be used in applications where the jumps have to be estimated immediately at the times when they occur and it is thus directly comparable with the L -estimator.

Overall, three jump estimation methods are compared for each of the simulated time series, with the Accuracy Ratio being used as the measure for their comparison. The tested methods are:

1. Non-parametric jump estimator based on the L -Statistics and local volatility, estimated with the bipower variation using the last 16 daily returns (as recommended by Lee and Mykland, 2008)
2. In-Sample Bayesian jump probability of occurrence estimates, based on a SVJD model with self-exciting jumps, estimated with 10 000 iterations of a MCMC algorithm, with the first 3 000 being discarded, while the later 7 000 are used to compute the Bayesian

- probabilities of jump occurrence by calculating for each day in the time series the average of the jump occurrences sampled in the iterations.
3. Out-Sample Bayesian jump probability of occurrence estimates, estimated by using a Particle Filter with 10 000 particles (i.e. computing the weighted average of the values of the jump and non-jump particles for each day), with model parameters and initial latent states at time zero estimated with a MCMC algorithm (again with 10 000 iterations and the first 3 000 discarded) that is run on simulated time series (with the same length as the original one), preceding the target time series.

In all of the tests, the target simulated time series consists of 5 000 trading days. This means that in order to be able to use the MCMC & SIR Particle Filter approach, a series of 10 000 trading days has to be simulated, with the first half representing the in-sample period (used for model estimation) and the second half the out-sample period (used for jump estimation and subsequent model comparison). The *L*-Estimator and the in-sample parametric model using MCMC are applied directly to the second half of the time series (i.e. to the last 5 000 observations, corresponding to the out-sample period). The third approach, combining MCMC with a SIR particle filter to construct out-sample jump forecasts, would first use the MCMC algorithm to estimate the model parameters and the evolution of its latent state variables on the first half of the simulated time series (in-sample). These parameters and the values of the latent state variables on last day of the in-sample period are then used to initialize the 10 000 particles of the SIR Particle Filter (by sampling from the latent state variables with repetition). The SIR Particle filter is then used to sequentially estimate the evolution of the latent state variables (i.e. the stochastic variances, jump occurrences and jump sizes) in the out-sample period.

As in the simulation study in section 3.1, the underlying jump process is simulated with 3 different kinds of dynamics of the jump intensity process. For each type of dynamics the series is additionally simulated in 10 variants differing in the magnitude of the jump sizes (i.e. with the parameter σ_{J} ranging from 0.005 to 0.05). The three types of jump intensity dynamics used in the simulations are:

- A. Jumps follow a Poisson process with constant jump intensity at the value $\lambda_{\text{LT}}=0.05$. The other parameters are set as in Table 1.
- B. Jumps follow a self-exciting Hawkes process with large persistence of the jump intensity ($\beta_{\text{J}}=0.98$) and small magnitude of jump self-excitation ($\gamma_{\text{J}}=0.015$). The other parameters are given as in Table 3.
- C. Jumps follow a self-exciting Hawkes process with low intensity persistence ($\beta_{\text{J}}=0.6$) and high magnitude of jump self-excitation ($\gamma_{\text{J}}=0.1$). This is the case that leads to co-jumps. Parameters used for simulating this variant are given in Table 5.

Table 7 shows the Accuracy Ratios of the 3 tested jump estimation methods when applied to the time series simulated with jumps following a Poisson process with constant jump intensity, as described in variant A. The Accuracy Ratios represent averages from 20 different runs of the simulation and estimation procedure. The averaging was performed in order for the results to not be overly influenced by the randomness of the simulations, so that we can draw reasonable conclusions about the accuracy of the tested methods.

Table 7 – Average Accuracy Ratios of the parametric and the non-parametric approach of jump identification applied to 20 simulations of the SVJD process with Poisson jumps

| | 0,005 | 0,01 | 0,015 | 0,02 | 0,025 | 0,03 | 0,035 | 0,04 | 0,045 | 0,05 |
|-------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| L-Estimator | 0,0270 | 0,1233 | 0,1939 | 0,3070 | 0,3843 | 0,4331 | 0,4909 | 0,5375 | 0,5685 | 0,6057 |
| MCMC | 0,0780 | 0,2342 | 0,3358 | 0,4814 | 0,5489 | 0,5995 | 0,6734 | 0,6955 | 0,7370 | 0,7580 |
| MCMC & SIR | 0,0118 | 0,0621 | 0,1700 | 0,2783 | 0,3768 | 0,4395 | 0,5000 | 0,5526 | 0,5712 | 0,6166 |

We can see from Table 7 that the in-sample MCMC estimation of the SVJD model leads to the most accurate jump estimates for all of the simulation variants (differing in the size of sigmaJ). When the approach of the task is to estimate the past jumps in a given financial time series ex-post, the SVJD & MCMC seems to be the most accurate method to perform the task. Among the two methods estimating the jump occurrences sequentially, immediately at the times at which the jumps occur, the non-parametric L-Estimator seems to provide slightly better results for the small jump sizes, while the parametric approach utilizing a SVJD model and a MCMC & SIR estimation procedure seems to be more accurate method for the large jumps sizes. The differences between the results of the two methods for large jump time series is however much smaller than for the small jump time series (especially in the case of the 1% jumps), indicating that the simple non-parametric L-Estimator may surprisingly be more suitable for real-world financial applications.

Table 8 shows the results of the tested jump identification methods when applied to a SVJD process with jumps following a self-exciting Hawkes process whose intensity only slowly decays toward its long-term level, causing the existence periods of high and low jump intensity (jump clusters).

Table 8 – Average Accuracy Ratios of the parametric and the non-parametric jump identification applied to 20 simulations of a SVJD process with persistent Hawkes jumps

| | 0,005 | 0,01 | 0,015 | 0,02 | 0,025 | 0,03 | 0,035 | 0,04 | 0,045 | 0,05 |
|-------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| L-Estimator | 0,0318 | 0,1153 | 0,2051 | 0,2726 | 0,3401 | 0,4321 | 0,4771 | 0,5110 | 0,5458 | 0,5754 |
| MCMC | 0,0783 | 0,2195 | 0,3636 | 0,4666 | 0,5384 | 0,6287 | 0,6862 | 0,7031 | 0,7257 | 0,7740 |
| MCMC & SIR | 0,0107 | 0,0561 | 0,1650 | 0,2679 | 0,3394 | 0,4586 | 0,5358 | 0,5595 | 0,5977 | 0,6658 |

It is apparent from the results that the in-sample MCMC estimation still remains the most accurate method of jump detection even in the case of the self-exciting Hawkes jumps with clustering effects. As for the other two methods, the difference between the results of the MCMC & SIR approach and the L-Estimator for the case of large jumps has increased when compared with the results for the large jumps in the case of the Poisson jumps simulations. Nevertheless, for small jumps, especially around 1% in absolute value, does the L-Estimator still provide better results than the parametric approach.

Finally, Table 9 shows the results for the simulations of a SJVD model with jumps governed by a Hawkes process with high value of the self-excitation parameter and high value of the jump intensity decay rate (1-betaJ), leading to the effect of co-jumps.

Table 9 – Average Accuracy Ratios of the parametric and the non-parametric jump identification applied to 20 simulations of a SVJD process with co-jumps

| | 0,005 | 0,01 | 0,015 | 0,02 | 0,025 | 0,03 | 0,035 | 0,04 | 0,045 | 0,05 |
|-------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| L-Estimator | 0,0325 | 0,1058 | 0,2065 | 0,2839 | 0,3664 | 0,4296 | 0,4640 | 0,5152 | 0,5466 | 0,5828 |
| MCMC | 0,0541 | 0,2018 | 0,3773 | 0,4730 | 0,5683 | 0,6303 | 0,6565 | 0,7179 | 0,7418 | 0,7740 |
| MCMC & SIR | 0,0177 | 0,0685 | 0,2022 | 0,2932 | 0,4016 | 0,4832 | 0,5293 | 0,5900 | 0,6299 | 0,6747 |

We can see that the results in Table 9, corresponding to the simulated time series with self-exciting effects leading to co-jumps, are similar as in the previous cases. The most accurate approach is again the in-sample estimation of the SVJD model with a MCMC algorithm, while the out-sample forecasts of the parametric model, constructed with the MCMC combined with a SIR Particle Filter are better than the non-parametric *L*-Estimator mostly for the simulations where the jumps are large (i.e. large sigmaJ), while for the smaller jumps, the *L*-Estimator seems to provide the better results again. Nevertheless, the differences between the two methods in the case of large jumps have further increased, and the differences in the case of small jumps further decreased, indicating that co-jumps dynamics slightly favours the parametric approach.

The overall conclusions from this section therefore are, that for in-sample analysis the parametric approach seems to outperform the non-parametric *L*-Estimator dramatically. For the purpose of sequential out-sample jump estimation, however, are the results of the non-parametric and the parametric approach suite similar, with the parametric approach (using SVJD estimated with MCMC and SIR) being more accurate when the jumps exhibit self-exciting behaviour and are large, while in the case of a Poisson process with small jumps, the *L*-Estimator does provide the better results out of the two methods.

4. Application of the methods to currency exchange rate time series

In this section, the parametric and the non-parametric jump estimation methods are applied to the time series of 4 major foreign exchange rates (EUR/USD, GBP/USD, USD/CHF and USD/JPY) and their results are compared. The analysed time series range from 1.11.1999 to 12.6.2015 and contain altogether 4 062 daily observations. The series were provided by forexhistorydatabase.com.

The methods compared in this section are the *L*-Estimator, applied with 16 days used in the calculation of the local volatility, and the in-sample application of the SVJD model, estimated with a MCMC algorithm with 10 000 iterations, the first 3 000 discarded and the remaining 7 000 used to calculate the estimates of model parameters and the latent states based on the posterior sample means. In the case of the jumps estimates, the MCMC means can be interpreted as Bayesian probabilities of jump occurrence at the given days. Additionally, a prior was used in the case of jump estimation, setting the value of sigmaJ to be on average three times larger than the unconditional volatility of the returns in the analysed time series. The reason for this is to set the prior jump distribution to be sufficiently high in order to avoid a situation in which jumps would be estimated to happen constantly on almost any day.

Table 10 shows the estimated values of the parameters and the Bayesian standard errors of the SVJD model estimated with MCMC.

Table 10 – Parameter estimates and Bayesian standard errors of the SVJD model estimated with a MCMC algorithm on the period from 1.11.1999 to 12.6.2015

| | | mui | muiJ | sigmaJ | alpha | beta | gamma | lambdaLT | betaJ | gammaJ |
|---------------|------------------|------------|-------------|---------------|--------------|-------------|--------------|-----------------|--------------|---------------|
| EURUSD | parameter | 0,0000 | 0,0009 | 0,0068 | -0,0500 | 0,9951 | 0,0661 | 0,0431 | 0,4523 | 0,0315 |
| | std.error | 0,0001 | 0,0018 | 0,0010 | 0,0191 | 0,0018 | 0,0079 | 0,0256 | 0,2733 | 0,0250 |
| GBPUSD | parameter | 0,0001 | -0,0006 | 0,0056 | -0,0744 | 0,9930 | 0,0757 | 0,0610 | 0,3584 | 0,0376 |
| | std.error | 0,0001 | 0,0018 | 0,0007 | 0,0229 | 0,0021 | 0,0078 | 0,0279 | 0,2402 | 0,0263 |
| USDCHF | parameter | -0,0001 | -0,0139 | 0,0551 | -0,0686 | 0,9932 | 0,0664 | 0,0027 | 0,6214 | 0,0666 |
| | std.error | 0,0001 | 0,0202 | 0,0146 | 0,0272 | 0,0027 | 0,0113 | 0,0016 | 0,2543 | 0,0257 |
| USDJPY | parameter | 0,0001 | -0,0009 | 0,0102 | -0,1642 | 0,9843 | 0,1166 | 0,0557 | 0,3970 | 0,0290 |
| | std.error | 0,0001 | 0,0012 | 0,0012 | 0,0405 | 0,0039 | 0,0107 | 0,0157 | 0,2469 | 0,0220 |

We can see from Table 10 that all of the analysed time series exhibit highly persistent stochastic volatility (beta in the log-variance model close to one), which is, however, in all of the cases stationary (beta is more than two standard errors away from one for all of the models). The highest persistence (beta) exhibits the stochastic volatility of the EUR/USD exchange rate, while the highest volatility of the volatility (gamma) exhibits the USD/JPY exchange rate, which is also the least persistent one, although its persistence is still very high.

Jumps in the time series are in most of the cases rather small, equal to approximately 0.5-1%, as seen from the parameter sigmaJ. The jumps also seem to occur quite frequently in the time series, with the long-term jump intensity (lambdaLT) being for most of the series around 5%, corresponding to approximately one jump per month. An exception in both of the cases (i.e. size and frequency of the jumps) is the USD/CHF exchange rate whose jumps were estimated to be on average equal to 5.5% in their size (sigmaJ), and to happen extremely uncommonly (0.27% daily intensity). We will see later that this was caused by the model getting fitted itself to the two extreme events in the time series, when the Swiss Central Bank started and subsequently ended its monetary interventions, which resulted in two massive jumps. In applications, these should probably be modelled rather as outliers or another jump distribution has to be used, with significantly thicker tails than the normal one.

As for the Hawkes process parameters, none of the time series exhibits long-term jump clustering in the analysed period, as the parameter betaJ is mostly low and statistically insignificant (except for USD/CHF for which it is low, i.e. much lower than one, but significant), nor do the series seem to exhibit significant co-jump behaviour, as the parameter gammaJ is also insignificant for most of the time series, again with the exception of USD/CHF, for which it is significant and moderately high, indicating certain co-jump behaviour (or possibly a very short-term jump clustering).

Figure 1 (in the appendix) shows the evolution of the stochastic variance estimates for all of the currency exchange rate time series. It is apparent that the series exhibit long memory as the volatility often drifts away from its long-term mean for many months. We can also see that all of the time series exhibited the greatest spike in volatility at the beginning of the year 2009, during the middle of the world financial crisis, with the CHF/USD experiencing a similar spike also during the 2011 Eurozone crisis, followed by the beginning of the monetary interventions by the Swiss Central Bank.

Figure 2 shows the Bayesian probabilities of jump occurrence. It is apparent that except for the USD/JPY, the other series contain only several jumps with Bayesian probability of occurrence higher than 50%. In the case of USD/JPY the higher frequency of jumps could then possibly be caused by the regular interventions of its central bank into its exchange rate. Additionally, we can see that the USD/CHF contains only 2 jumps, both with probability of occurrence close to 100%, and basically no

other jumps apart from that. This indicates that the model has fitted itself to the two extreme jumps associated with the start and the end of the interventions of the Swiss Central Bank, causing the model to not be able to model the other jumps efficiently, as these two extreme events caused the estimated sigmaJ to be too large. As already mentioned, to solve this problem, it is possible to either treat the two extreme observations as outliers, or to utilize a different jump size distribution, with far fatter tails than the normal one, that would allow the model to model standard as well as extreme jumps as in the case of the USD/CHF exchange rate.

Finally, it is worth noting that the jumps surprisingly do not increase in frequency during the world financial crisis in 2009, which might be counterintuitive. We believe this to be caused by the fact that the jump volatility (sigmaJ) is assumed to be constant over time, while the continuous price volatility is not. This caused the model to assign all of the large jump movements during the crisis period rather to the continuous volatility than to the jump component. To alleviate this issue it would be possible to work with time-varying jump volatility, which would be an interesting extension to be tested in further studies, in order to assess whether this could improve jump estimation through the SVJD model or not.

Figure 3 shows the probabilities of jump occurrence inferred from the non-parametric *L*-Estimator applied to the four analysed currency exchange rate time series. We can see that the estimator identifies significantly more jumps with very high probabilities of occurrence (i.e. above 90%). Additionally, the jumps seem to be uniformly spread through the time series, including the period during the Global Financial Crisis at the beginning of year 2008, unlike the parametrically identified jumps which seem to occur far less frequently during this period. The reason for this is the different approach to jump identifications, determining whether a return was a jump or not, solely based on how large it is compared to the current local volatility, without the use of any assumptions about the distribution of jump sizes and their average volatility (i.e. absolute size). As it seems, this gives the *L*-Estimator higher power in the case of time-varying volatility of the underlying price process, compared to the SVJD approach in which the jump distribution is held constant.

While the parametric and the non-parametric jump occurrence probabilities seem to differ significantly in their size for all of the time series, it is still possible to ask, whether they provide at least a similar information with regards to their discriminatory power between jump days and non-jump days. Specifically, we could ask, whether if the parametric approach assigns to a certain day a relatively high jump probability (compared to the other days), does the non-parametric approach assign to that day a relatively high jump probability as well. In order to answer this, rank correlation between the jump probabilities of the two methods was computed for all of the time series (specifically for the *L*-Estimator the values of the test statistics are used instead of the inferred probabilities of jump occurrence as these are in vast majority of days either 0 or 1, so the strength of the prediction cannot be assessed). We compute the standard linear Pearson correlation as well as the Spearman rank correlation. The results are shown in Table 11.

Table 11 – Pearson and Spearman correlations between the parametric and the non-parametric jump estimates

| | EURUSD | GBPUSD | USDCHF | USDJPY |
|----------|--------|--------|--------|--------|
| Pearson | 0,6659 | 0,7245 | 0,4898 | 0,7497 |
| Spearman | 0,7471 | 0,7746 | 0,4843 | 0,7554 |

We can see from Table 11 that the correlations between the two jump estimators are relatively high (70-80%) for all of the time series, except for USD/CHF for which the correlation is slightly lower. We

believe this to be caused by the already mentioned effect of the Swiss Central Bank interventions on the estimation of the SVJD model parameters, influencing the jump estimates.

Finally, we use the calculated jump estimates in order to asset to what degree do the jumps coincide between the 4 analysed foreign exchange time series. For this purpose, Pearson correlation is computed between the jump estimates for all of the series, separately for the parametric and the non-parametric case.

Table 12 – Correlations between the jump occurrences of the different currency foreign exchange rate time series

| | | EURUSD | GBPUSD | USDCHF | USDJPY |
|--|---------------|---------------|---------------|---------------|---------------|
| Parametric SVJD jump estimates | EURUSD | 1,0000 | 0,3180 | 0,0971 | 0,1417 |
| | GBPUSD | 0,3180 | 1,0000 | 0,0258 | 0,1302 |
| | USDCHF | 0,0971 | 0,0258 | 1,0000 | 0,0214 |
| | USDJPY | 0,1417 | 0,1302 | 0,0214 | 1,0000 |
| <i>L</i> -Estimator based jump estimates | | EURUSD | GBPUSD | USDCHF | USDJPY |
| | EURUSD | 1,0000 | 0,4216 | 0,6086 | 0,2143 |
| | GBPUSD | 0,4216 | 1,0000 | 0,2793 | 0,2123 |
| | USDCHF | 0,6086 | 0,2793 | 1,0000 | 0,1961 |
| | USDJPY | 0,2143 | 0,2123 | 0,1961 | 1,0000 |

By comparing the results in the two tables, we see that they give generally the same information for all of the currencies except for the USD/CHF, which further contributes to the idea that the parametric jump estimates for this currency are biased. As for the conclusions of the analysis, there seems to be a relatively strong correlation between the jumps in the EUR/USD and USD/CHF (when measured with the *L*-Estimator) and further also between the EUR/USD and the GBP/USD, while the correlation between the JPY and the other currencies seems to be slightly lower.

Conclusion

In the study we compared the accuracy of a parametric and a non-parametric approach to jump identification on simulated as well as real-world daily frequency foreign exchange rate time series. As a non-parametric approach, *L*-Estimator was used, estimating jumps based on returns normalized with local volatility estimated with bipower variation. As a non-parametric approach a Stochastic-Volatility Jump-Diffusion (SVJD) model was used, whose parameters and latent state variables (stochastic variances, jump occurrences and jump sizes) were estimated with Bayesian methods. Specifically, a MCMC algorithm was used to estimate the latent time series of jump occurrences in the in-sample period, while a combination of MCMC and a Sequential Importance Resampling (SIR) Particle Filter was used to estimate jumps out-sample, immediately at the times at which they occur.

The first part of the simulation study confirmed that the value of number of days $k=16$ (recommended by Lee and Mykland 2008) used in the *L*-Estimator for local volatility calculation is close to optimal, except for the cases when the simulated jumps are very small (0.5%), in which case lower values would be preferable. The study also showed that if the simulated time series contains co-jumps then generally larger values of k may be preferable, nevertheless, the increase of performance when compared to the value of $k=16$ was rather low.

In the second part of the simulation study, the *L*-Estimator was compared with the parametric in-sample MCMC estimation of jumps in the time series as well as with the approach utilizing MCMC together with a SIR Particle Filter to generate out-sample sequential jump estimates. The study showed that the in-sample MCMC estimation of a SVJD model provides by far the most accurate jump forecasts, while among the two methods that estimate the jumps sequentially, immediately at the times at which they occur (i.e. the *L*-Estimator and the method utilizing MCMC and a SIR Particle Filter), the results were inconclusive. Specifically, for the case of independent Poisson jumps, the *L*-Estimator achieved higher accuracy in the case of small jumps and comparable accuracy to the parametric approach in the case of large jumps. In the cases when either long-term jump clustering or co-jumps in the simulated time series were assumed, the parametric approach outperformed the *L*-Estimator in the case of the larger jumps, nevertheless, in the case of small jumps (1% and less), the *L*-Estimator still provided better results than the parametric method.

In the last part of the study, *L*-Estimator and the parametric approach based on a in-sample MCMC estimation of a SVJD model, were applied to the daily time series of 4 major currency exchange rates, namely to EUR/USD, GBP/USD, USD/CHF and USD/JPY. Both of the methods identified certain jumps in all of the time series, the parametric approach did, however, identify relatively less jumps during the crisis period of 2009, while the non-parametric approach did not. This indicates that the non-parametric approach might potentially be more robust to changing levels of market volatility, as it does not assume any distribution for the jump sizes, while the SVJD model does, in our case, assume a distribution with fixed jump volatility, which may complicate the identification of jumps in periods when the volatility is high. This makes the parametric approach also more sensitive to outliers. This had the effect that only 2 jumps were identified in the USD/CHF exchange rate, corresponding to the start and the end of the Swiss Central Bank monetary interventions, when two extreme jumps occurred, causing the estimated jump distribution to get fitted solely to them.

Finally, correlation between the jump probability estimates of the two methods was compared, indicating relatively high correlation, with the exception of the USD/CHF rate for which the parametric approach behaved anomalously as mentioned. Correlation was additionally examined between the jump probabilities in different time series, indicating that jumps are correlated strongly between the CHF/USD and EUR/USD and further between EUR/USD and the GBP/USD.

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Appendix – MCMC and L-Estimator estimate charts

Figure 1 – Stochastic volatility estimates from the SVJD model estimated with MCMC

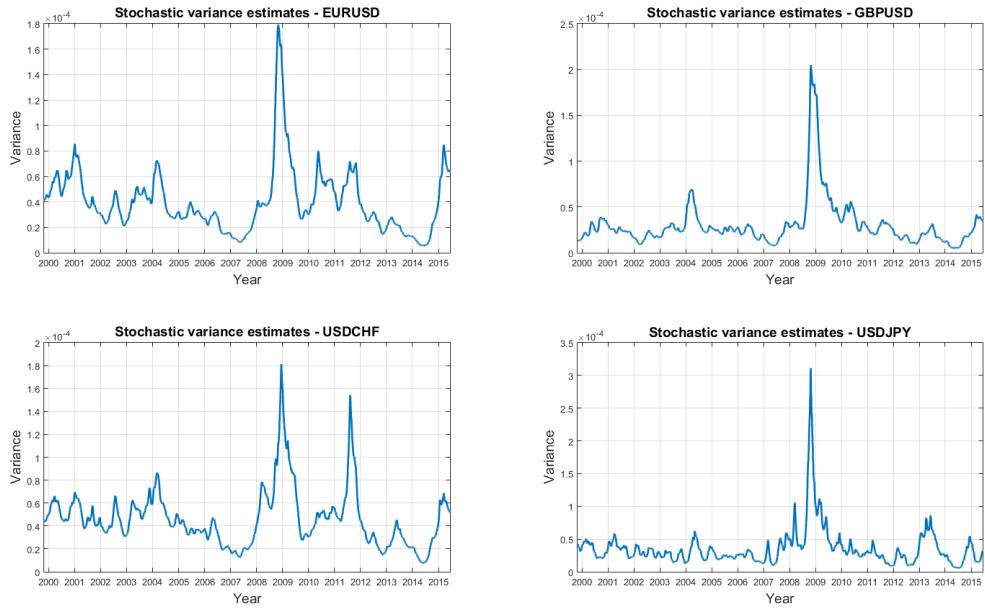


Figure 2 – Bayesian jump probabilities estimated with MCMC

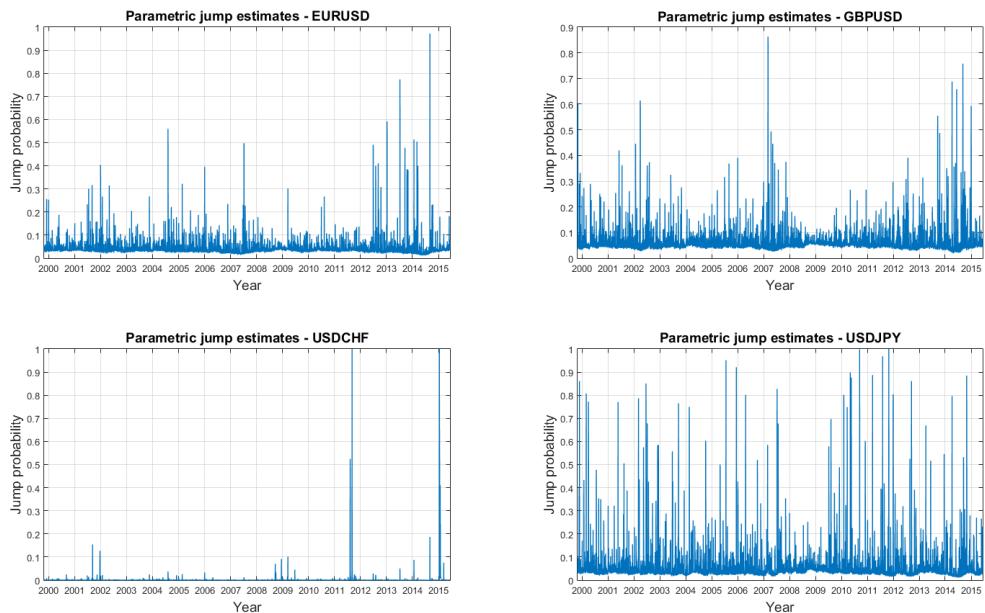


Figure 3 – Jump probabilities estimated with the non-parametric *L*-Estimator

