

GIBBSIAN PROCESSES OF CONVEX GRAINS

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Abstract: Gibbsian process of grains is usually constructed as marked point process with grains situated in its points, where the distribution of grains is deterministic or comes from some family of distributions. We construct a model directly from the definition of a pair potential on pairs of bounded sets in \mathbb{R}^d . The method is equivalent to that used in Preston [6].

Keywords: Gibbsian point process, pair potential

1 Introduction

Processes of mutually non-overlapping grains appear in many works. Their theoretical construction is known in two different ways. One approach starts with a Poisson point process in whose points convex grains are situated. By different kinds of thinning, using for example assignment of weights to the pairs of overlapping points, from the Poisson germ grain model the model of non-overlapping grains is derived. It means that some of the grains are removed from the model which causes that the distribution of the resulting process and the grain distribution changes. These thinning procedures can be found in [2].

A more elegant approach defines directly a non-overlapping model. Its idea is based on the well-known hard-core process and its generalization, the Gibbsian process. The theory of Gibbsian processes deals with the energy of the system of particles and could be easily fitted to this problem. A model of non-overlapping circles is defined by putting infinite energy to configurations where some of the circles overlap. Mase ([3]) studied how changes the particle diameter distribution in case of general hard-core process of balls.

This method can be generalized from processes of random diameter circles to processes of arbitrary convex shapes by defining energy and hence Gibbsian processes on them.

2 The Gibbsian Process

First, we will define general Gibbsian process on a locally compact, second countable Hausdorff space X . The definition is based on the theory of specifications, which can be found in [6].

Let \mathcal{B} be the σ -field of Borel sets in X and let \mathcal{B}_0 denote all bounded sets in \mathcal{B} (i.e. sets with compact closure).

A point process Φ on X is a measurable mapping from some probability space $(\Omega, \mathcal{A}, \Pr)$ into $(\mathbb{N}, \mathcal{N})$, the space of locally finite integer-valued measures on X , while the σ -field \mathcal{N} is defined that for all $B \in \mathcal{B}$, $\mu \mapsto \mu(B)$ is measurable, $\mu \in \mathcal{N}$. We will suppose that the measure of each atom is at most one, that is

$$\mu \in \mathbb{N} \Rightarrow \mu(x) \leq 1 \quad \forall x \in X. \quad (1)$$

We will denote the distribution of Φ by $Q = \Pr \Phi^{-1}$ and $\Lambda = E\Phi$ is the intensity measure.

For $G \in \mathcal{B}$ the sub- σ -field \mathcal{N}_G of \mathcal{N} consists of events occurring in G , i.e. it is generated by functions $\mu \mapsto \mu(G')$, $G' \subseteq G$.

Definition: Let $R_G \in \mathcal{N}_{G^c}$, $G \in \mathcal{B}_0$. A collection of probability kernels $\{\Pi_G\}_{G \in \mathcal{B}_0}$ is called a *specification* with respect to $\{R_G\}_{G \in \mathcal{B}_0}$, iff

- i) $\Pi_G(\cdot, B)$ is \mathcal{N}_{G^c} -measurable for all $G \in \mathcal{B}_0, B \in \mathcal{N}$
- ii) $\Pi_G(x, B) = 0$ for all $x \notin R_G, G \in \mathcal{B}_0, B \in \mathcal{N}$
- iii) $\Pi_G(\cdot, B) = I_{B \cap R_G}$ for all $B \in \mathcal{N}_{G^c}, G \in \mathcal{B}_0$
- iv) $\Pi_G \Pi_{G'} = \Pi_G$ whenever $G' \subseteq G$.

Definition: Let $\Pi = \{\Pi_G\}_{G \in \mathcal{B}_0}$ be a specification. The set $G(\Pi)$ of *Gibbsian processes* consists of all Φ , for which

$$\Pr \left(\Phi \in B \mid \Phi|_{G^c} \right) = \Pi_G(\cdot, B) \quad \text{a.s.} \quad (2)$$

If the specification Π fulfills certain conditions then the set $G(\Pi)$ is non-void ([6], Theorem 3.1.).

One standard approach how to define a specification (sometimes it is also called local specification) is to deal with potentials and local energy of the system of particles. The following method is similar to that used in [5], [6]. We will denote by \mathbb{N}_f the set of all finite measures from \mathbb{N} ($\mu \in \mathbb{N}_f \Rightarrow \mu(X) < \infty$) and by \mathbb{N}_G the set of all measures with support in $G \in \mathcal{B}_0$.

There is a natural order on \mathbb{N} where for $\mu, \nu \in \mathbb{N}$, $\mu \leq \nu \Leftrightarrow \mu$ is a sub-configuration of ν (there exists $\omega \in \mathbb{N}, \mu + \omega = \nu$).

The potential is a measurable function $V : \mathbb{N}_f \rightarrow (-\infty, \infty]$ such that

$$V(0) = 0, \quad (3)$$

$$V(\mu) = \infty \Rightarrow V(\nu) = \infty, \quad \mu \leq \nu. \quad (4)$$

We suppose that V can be written in the form

$$V(\nu) = \sum_{\mu \leq \nu} U(\mu), \quad (5)$$

where again $U : N_f \rightarrow (-\infty, \infty]$ and $U(0) = 0$. U is called the interaction potential. Sometimes the simpler form of U is considered, assuming that $U(\mu) = 0$ if $\mu(X) \neq 2$. Then U is called a *pair potential*.

To define a specification we will proceed as follows. For $G, G' \in \mathcal{B}_0$ with $G \subseteq G'$, define, $E_{G,G'} : N \rightarrow (-\infty, \infty]$ by

$$E_{G,G'}(\mu) = \sum \{V(\nu) : \nu \leq \mu|_{G'}, \nu(G) > 0\}, \quad (6)$$

where the sum over the empty set is taken to be zero. Set

$$\begin{aligned} R_G^0 &= \{\mu \in N : \{E_{G,G'}(\mu)\}_{G' \in \mathcal{B}_0} \text{ converges absolutely if } G' \nearrow X\} \cup \\ &\cup \{\mu : E_{G,G'}(\mu) = \infty \text{ for some } G' \in \mathcal{B}_0\} \end{aligned}$$

and

$$E_G(\mu) = \lim_{G' \nearrow X} E_{G,G'}(\mu) \text{ for } \mu \in R_G^0. \quad (7)$$

The system of regular subsets $\{R_G\}_{G \in \mathcal{B}_0}$ of N is defined as

$$\begin{aligned} R_G &= \{\mu \in N : \xi + \mu|_{X \setminus G} \in R_G^0 \text{ for all } \xi \in N_G\} \cap \\ &\cap \{\mu \in N : \exists c \in \mathbb{R}^- \forall \xi \in N_G; E_G(\xi + \mu|_{X \setminus G}) \geq \xi(G) \cdot c\} \cap \\ &\cap \{\mu \in N : V(\nu) < \infty \text{ for all } \nu \in N_f, \nu \leq \mu|_{X \setminus G}\}. \end{aligned}$$

Suppose that the potential V is *stable*, i.e. there exists $K \in \mathbb{R}^-$, such that $V(\mu) \geq \mu(X) \cdot K$.

Let Q_P^G be the distribution of some Poisson process on X restricted to $G \in \mathcal{B}_0$ and set

$$Z(G, \mu) = \int_{N|_G} \exp\{-E_G(\xi + \mu|_{G^c})\} Q_P^G(d\xi). \quad (8)$$

V being stable guarantees that $Z(G, \mu) < \infty$. If we define

$$\begin{aligned} \Pi_G(\mu, B) &= \frac{1}{Z(G, \mu)} \int_{\{\xi \in N|_G : \xi + \mu|_{G^c} \in B\}} \exp\{-E_G(\xi + \mu|_{G^c})\} Q_P^G(d\xi) \quad \text{if } \mu \in R_G \quad (9) \\ &= 0 \quad \text{otherwise,} \quad (10) \end{aligned}$$

then $\{\Pi_G\}_{G \in \mathcal{B}_0}$ is a specification with respect to $\{R_G\}$. The stability of the interaction V also guarantees that $R_G, G \in \mathcal{B}_0$ are non void (Proposition 6.3., [6]).

A computation of characteristics of Gibbsian processes is difficult, even, in many cases it is impossible. A simulation is one way how to derive some estimates. We have used a *Metropolis-Hastings algorithm* (described in [7]) to make some examples in the last section.

Another interesting way might be using the approximation of mean characteristics which developed Mase in [4]. The result of his work is the expression of an expectation of the form

$$E \left\{ \prod_{x \in \mu} g(x) \right\}, \quad (11)$$

where μ is a Gibbsian process and g is a function which is bounded in certain way.

The expectation is shown to be exponential of a polynomial in a characteristic z called *activity* (in [8] $-\log z$ is called a *chemical potential*, in this paper z is corresponding to the intensity of a reference Poisson process).

2.1 Exterior conditioning

The theory of local specifications is connected to a method described in [1], chapter 14, exterior conditioning.

Suppose δ_x is a Dirac measure, i.e.

$$\delta_x(A) = 1 \quad x \in A, \quad (12)$$

$$= 0 \quad \text{otherwise.} \quad (13)$$

If for point process Φ with a distribution Q holds

$$\forall B \in \mathcal{B}_0 \quad \Pr[\Phi(B) = 0 \mid \Phi|_{B^c}] > 0 \quad \text{a.s.} \quad (14)$$

then its modified Campbell measure $C^!$, defined by

$$\int \int f(x, \mu) C^!(d(x, \mu)) = \int \int f(x, \mu - \delta_x) \mu(dx) Q(d\mu) \quad \forall f \text{ measurable on } X \times \mathbb{N} \quad (15)$$

can be disintegrated in the form

$$\int f(x, \mu) C^!(d(x, \mu)) = \int \int f(x, \mu) \lambda(dx|\mu) Q(d\mu), \quad (16)$$

where the probability kernel $\lambda(\cdot|\mu)$ is called Papangelou kernel or conditioned intensity of the process Φ .

If we apply this result to a Gibbsian process with a pair potential U and a *local energy* $E(x|\mu)$ defined as

$$E(x|\mu) = \sum_{y \in \mu} U(x, y), \quad (17)$$

we derive the statement of the next theorem.

Theorem ([5]): Let Φ be a Gibbsian process with the distribution Q , conditional energy $E(x|\mu)$ and reference Poisson process Φ_P with the intensity measure ρ . Then for all nonnegative measurable functions h on $(X, \mathcal{B}) \times (\mathbb{N}, \mathcal{N})$

$$\int_{\mathbb{N}} \int_X h(K, \mu) \mu(dx) Q(d\mu) = \int_{\mathbb{N}} \int_X h(K, \mu + \delta_x) \exp\{-E(x|\mu)\} \rho(dx) Q(d\mu).$$

3 Processes in $\mathcal{K}'(\mathbb{R}^d)$

Consider the space $\mathcal{K}'(\mathbb{R}^d)$ (or simply \mathcal{K}') of compact nonempty subsets of \mathbb{R}^d . The Hausdorff metric on \mathcal{K}' is defined as follows

$$d_H(K, L) = \max \left\{ \sup_{x \in K} d(x, L), \sup_{y \in L} d(y, K) \right\},$$

where $d(a, A) = \inf_{b \in A} \varrho(a, b)$ and ϱ is the euclidean metric on \mathbb{R}^d . $(\mathcal{K}'(\mathbb{R}^d), d_H)$ is then locally compact, complete and separable.

We would like to define a specification Π on $\mathcal{N}(\mathcal{K}')$ and show that the set $G(\Pi)$ is not empty.

Consider the pair potential U on $\mathcal{K}' \times \mathcal{K}'$ in the form

$$\begin{aligned} U(K, L) &= \infty, & \text{if } d_H(K, L) = 0 \\ &= \frac{(\lambda^d(K))^2 \cdot (\lambda^d(L))^2}{d_H(K, L)}, & \text{otherwise,} \end{aligned}$$

where λ^d indicates d -dimensional Lebesgue measure. Thus the corresponding potential $V(\mu) = \sum_{K, L \in \mu} U(K, L)$ is stable because $U(K, L) \geq 0 \forall K, L \in \mathcal{K}_0$.

Let $\mu \in \mathcal{N}(\mathcal{K}')$, $K \in \mathcal{K}'$. The conditional energy of K given μ is

$$E(K|\mu) = \sum_{L \in \mu} U(K, L).$$

Now we can define the total energy of a bounded borel set $G \in \mathcal{B}_0(\mathcal{K}')$ with respect to a configuration μ by

$$E_G(\mu) = \sum_{K \in \mu|_G} E(K|\mu) + \sum_{K, L \in \mu|_G, K \neq L} U(K, L).$$

Let $B \in \mathcal{N}(\mathcal{K}')$, $G \in \mathcal{B}'_0$, $\mu \in \mathcal{N}$ and let

$$\Pi_G(\mu, B) = \frac{1}{Z(G, \mu)} \int_{\{\xi \in \mathcal{N}|_G: \xi + \mu|_{G^c} \in B\}} \exp\{-E_G(\xi + \mu|_{G^c})\} Q_P^G(d\xi),$$

where $Z(G, \mu) = \int_{\mathcal{N}|_G} \exp\{-E_G(\xi + \mu|_{G^c})\} Q_P^G(d\xi)$ is a norming constant and Q_P^G the distribution of some reference Poisson process on $\mathcal{K}'(\mathbb{R}^d)$, restricted to G . The system $\{\Pi_G(\mu, A)\}_{G \in \mathcal{B}_0, \mu \in \mathcal{N}, A \in \mathcal{N}}$ is then a specification.

4 Examples of processes of nonoverlapping grains

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