A Copula Approach to CVA Modeling[1]

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Abstract

We consider counterparty credit risk in the interest rate swap (IRS) contracts in the presence of an adverse dependence between the default time and interest rates, so-called wrong-way risk. The IRS credit valuation adjustment (CVA) semi-analytical formula based on Gaussian copula assumption, presented in Černý and Witzany [2014], is further replaced by Fréchet copula (for extreme dependence) mainly based on the work of Cherubini [2013], called modified approach. The result of all three CVA calculation approaches are compared in a numerical study where we find that our semi-analytical formulas (the Gaussian copula and modified approach) provide more accurate information on IRS CVA price.

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1 Introduction

Among practitioners there are currently used two main methods of interest rate swap (IRS) credit valuation adjustment (CVA) calculation with inclusion of the wrong-way risk (WWR) - copula and simulation approach.

The copula approach aims to simplify the calculation of IRS CVA using analytical or semi-analytical formulas (see, e.g., Cherubini [2013]). On the other hand, the simulation approach is trying to faithfully model the behavior of real interest rates and intensity of default (see, e.g., Brigo and Pallavicini [2007] or Hull and White [2011]).

Both methods have their pros and cons. The advantage of the simulation approach is straightforward calibration of the model while the disadvantage is difficult implementation, time-consuming calculation and counterintuitive interpretation of the correlations. On the contrary, the copula approach allows quick calculation, easy implementation, and intuitive interpretation correlations. The disadvantage of the copula approach is the difficult calibration of the model and usage of simplifying (sometimes not realistic) assumptions.

In this paper, we present different copula approaches to IRS CVA calculation with inclusion of the WWR.

The second section contains general copula framework for contingent swaption pricing using Sklar's theorem. The Sorensen and Bollier [1994] is derived based on independent copula assumption.

In the third section we remind the risky swaption price and the IRS CVA semi-analytical formula with Gaussian copula assumption introduced in Černý and Witzany [2014].

Another IRS-CVA semi-analytical formula based on Fréchet copula introduced in Cherubini [2013] is outlined in the fourth section.

Since Cherubini's derivation of the semi-analytical formula is not completely consistent with the theory of copulas (particularly with the Sklar's theorem) we present a correction named as modified approach in the sixth section.

In the seventh section is the illustrative numerical study where we show that the CCR has a relevant impact on the IRS prices and that the wrong-way risk has a relevant impact on the CVA. We compare the results of our semi-analytical formulas (the Gaussian copula and modified approach). We also compare them with the results of Cherubini's approach.

We discuss the results, findings, and possible challenges in more detail in the conclusion.

2 General Copula Approach

Consider a contingent swaption with exercise at the time T_1, n swap payments at times $T_1 < \cdots < T_n = T$, fix paying rate s_{K}, that can be exercised only if the counterparty defaults until $\widetilde{T} \in (0, T_1]$. Then the price of such contingent swaption

V_{CS} can be expressed as

$$V_{\text{CS}}\left(0, \widetilde{T}, T_1, T_n, \omega\right) = X(0, T_1, T_n) \cdot \mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(\omega(s_{1,n}(T_1) - s_K))^+ \mathbb{1}\left[\tau \le \widetilde{T}\right]\right] \quad (2.1)$$

where ω stands for the side of the contract (i.e., $\omega = 1$ for the fix rate payer and $\omega = -1$ for the fix rate receiver), $\mathbb{1}\left[\tau \le \widetilde{T}\right]$ denotes the indicator function (i.e., it is equal to 1 if $\tau \le \widetilde{T}$ and 0 otherwise), X is the annuity defined as

$$X(t, T_1, T_n) = \sum_{i=1}^{n} \delta_i P(t, T_i) \quad (2.2)$$

δ_i is year fraction between payment dates T_{i-1} and T_i, $T_0 = 0$, dependent on the calendar convention, $P(t, T_i)$ is the price of a unit nominal zero-coupon bond at time t maturing at T, and $s_{1,n}(t)$ is the forward swap interest rate at time t starting at time T_1 with the first payment at T_2 (maturity of the swap is still T_n) is defined as

$$s_{1,n}(t) = \frac{P(t, T_1) - P(t, T_n)}{\sum_{k=2}^{n} \delta_k P(t, T_k)}. \quad (2.3)$$

The conditional expectation on the right-hand side of (2.1) can be rewritten, according to Cherubini [2013], as

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_{1,n}\ (T_1) - s_K)^+ \mathbb{1}\left[\tau \le \widetilde{T}\right]\right] = \\
&\qquad = \mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[\int_0^{\infty} \mathbb{1}\left[(s_{1,n}(T_1) - s_K)^+ > s\right] \mathbb{1}\left[\tau \le \widetilde{T}\right] \mathrm{d}s\right] \\
&\qquad = \mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[\int_{s_K}^{\infty} \mathbb{1}\left[s_{1,n}(T_1) > s\right] \mathbb{1}\left[\tau \le \widetilde{T}\right] \mathrm{d}s\right] \\
&\qquad = \int_{s_K}^{\infty} \mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[\mathbb{1}\left[s_{1,n}(T_1) > s\right] \mathbb{1}\left[\tau \le \widetilde{T}\right]\right] \mathrm{d}s \\
&\qquad = \int_{s_K}^{\infty} \mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[\mathbb{1}\left[s_{1,n}(T_1) > s, \tau \le \widetilde{T}\right]\right] \mathrm{d}s \\
&\qquad = \int_{s_K}^{\infty} \mathbb{Q}_{1,n}\left[s_{1,n}(T_1) > s, \tau \le \widetilde{T}\right] \mathrm{d}s
\end{aligned} \quad (2.4)$$

and analogously for the other side of the contract

$$\mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_K - s_{1,n}(T_1))^+ \mathbb{1}\left[\tau \le \widetilde{T}\right]\right] = \int_0^{s_K} \mathbb{Q}_{1,n}\left[s_{1,n}(T_1) \le s, \tau \le \widetilde{T}\right] \mathrm{d}s. \quad (2.5)$$

Let $\mathbb{C}$ be an arbitrary copula for the swap rate $s_{1,n}(T_1)$ and the default time τ, i.e.,

$$\mathbb{Q}_{1,n}\left[s_{1,n}(T_1) \leq s, \tau \leq \widetilde{T}\right] = \mathbb{C}(G(s), F(\widetilde{T})) \tag{2.6}$$

where G and F are the cumulative distribution functions of the swap rate $s_{1,n}(T_1)$ and the default time τ. Since the inequalities in (2.4) are not same we may use the copula $\widetilde{\mathbb{C}}(1-u, v)$ corresponding to the joint event of the swap rate being higher than the strike rate s_K and the default time of the counterparty being lower than time $\widetilde{T}$. Notice that copulas $\mathbb{C}$ and $\widetilde{\mathbb{C}}$ are different but the following equality holds $\widetilde{\mathbb{C}}(1-u, v) = v - \mathbb{C}(u, v)$. Then

$$\mathbb{Q}_{1,n}\left[s_{1,n}(T_1) > s, \tau \leq \widetilde{T}\right] = \widetilde{\mathbb{C}}(\overline{G}(s), F(\widetilde{T})) = F(\widetilde{T}) - \mathbb{C}(G(s), F(\widetilde{T})) \tag{2.7}$$

where $\overline{G}(x) = 1 - G(x)$.

Therefore, we may conclude that

$$\mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_{1,n}(T_1) - s_K)^+ \mathbb{1}\left[\tau \leq \widetilde{T}\right]\right] = \int_{s_K}^{\infty} \widetilde{\mathbb{C}}(\overline{G}(s), F(\widetilde{T}))ds \tag{2.8}$$

and

$$\mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_K - s_{1,n}(T_1))^+ \mathbb{1}\left[\tau \leq \widetilde{T}\right]\right] = \int_{0}^{s_K} \mathbb{C}(G(s), F(\widetilde{T}))ds. \tag{2.9}$$

For example, let us consider an independent copula

$$\mathbb{C}(u, v) = uv, \quad 0 \leq u \leq 1,\ 0 \leq v \leq 1, \tag{2.10}$$

that is, independence between default time and the swap rate, which should give us IRS CVA formula presented in Sorensen and Bollier [1994]. The fix-payer contigent swaption is calculated as

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_{1,n}(T_1) - s_K)^+ \mathbb{1}\left[\tau \leq \widetilde{T}\right]\right] &= F(\widetilde{T}) \int_{s_K}^{\infty} (\overline{G}(s))\mathrm{d}s \\
&= F(\widetilde{T}) \int_{s_K}^{\infty} \mathbb{Q}_{1,n}\left[s_{1,n}(T_1) > s\right] \mathrm{d}s \\
&= F(\widetilde{T}) \cdot \mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_{1,n}(T_1) - s_K)^+\right] \\
&= \frac{F(\widetilde{T}) V_{\text{Swaption}}(0, T_1, T_n, s_K, 1)}{X(0, T_1, T_n)}
\end{aligned} \tag{2.11}$$

and analogously for the fix-rate receiver contingent swaption we get

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_{\mathrm{K}}-s_{1,n}(T_1))^+\mathbb{1}\left[\tau\le\widetilde{T}\right]\right] &= F(\widetilde{T})\int_0^{s_{\mathrm{K}}} G(s)\mathrm{d}s \\
&= F(\widetilde{T})\int_0^{s_K} \mathbb{Q}_{1,n}\left[s_{1,n}(T_1)\le s\right]\mathrm{d}s \\
&= F(\widetilde{T})\cdot\mathbb{E}_0^{\mathbb{Q}_{1,n}}\left[(s_{\mathrm{K}}-s_{1,n}(T_1))^+\right] \\
&= \frac{F(\widetilde{T})V_{\mathrm{Swaption}}(0,T_1,T_n,s_{\mathrm{K}},-1)}{X(0,T_1,T_n)}. \quad (2.12)
\end{aligned}$$

Hence, after division of the time interval $(t,T]$ into disjoint subintervals $(T_0,T_1]$, $(T_1,T_2]$, ..., $(T_1,T_n]$ we obtain

$$\begin{aligned}
&\mathrm{ICVA}_{\mathrm{IRS}}(t,T) = \\
&\quad= \mathrm{LGD}\sum_{i=0}^{n-1}\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[T_i<\tau\le T_{i+1}\right]D(t,\tau)V(\tau,T)^+\right] \\
&\quad\approx \mathrm{LGD}\sum_{i=0}^{n-1}\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[T_i<\tau<T_{i+1}\right]D(t,T_{i+1})V(T_{i+1},T)^+\right] \\
&\quad= \mathrm{LGD}\sum_{i=0}^{n-1}(F(T_{i+1})-F(T_i))X(t,T_{i+1},T_n)\mathbb{E}_t^{\mathbb{Q}_{i+1,n}}\left[(\omega(s_{i+1,n}(T_{i+1})-s_{\mathrm{K}}))^+\right] \\
&\quad= \mathrm{LGD}\sum_{i=0}^{n-1}(F(T_{i+1})-F(T_i))V_{\mathrm{Swaption}}(t,T_{i+1},T_n,s_{\mathrm{K}},\omega),
\end{aligned} \quad (2.13)$$

which is equal to the Sorensen-Bollier approximation IRS CVA price. Both formulas depend on the contract side expressed by the last parameter ω.

3 Gaussian Copula Approach

In the paper Černý and Witzany [2014] we have shown that the interest rate swap (IRS) CVA can be approximated by swaption prices even in case that exposure and default time are not independent, i.e., with the inclusion of wrong-way, or right-way, risk. Let us remind the partial result of the paper which is the calculation of the risky swaption price given by the following theorem.

Theorem 1. *Suppose that U, ε_1, ε_2 are N(0,1) iid, $T_1>0$, $\sigma>0$, $a\in[-1,1]$, $b\in[-1,1]$, $S(t)=e^{-ht}$, and the swap rate $s_{1,n}(T_1)$ is expressed (with respect to the* *annuity measure $\mathbb{Q}_{1,n}$) as*

$$s_{1,n}(T_1) = s_{1,n}(0)\exp\left\{-\sigma^2 T_1/2 + \sigma\sqrt{T_1}Y\right\} \tag{3.1}$$

where $Y = aU + \sqrt{1-a^2}\varepsilon_1$. Further let us suppose that the default time is defined as

$$\tau = S^{-1}(\Phi(-Z)) \tag{3.2}$$

where $Z = bU + \sqrt{1-b^2}\varepsilon_2$. Then the risky payer (receiver) swaption price with strike rate s_{K}, no recovery, $0 < \widetilde{T} \leq T_1$, annuity numeraire satisfying (2.2) and with the payoff function $V_{\mathrm{RS}}^{\mathrm{payoff}} \equiv V_{\mathrm{RS}}(T_1, \widetilde{T}, T_1, T_n)$ equal to

$$V_{\mathrm{RS}}^{\mathrm{payoff}} = \begin{cases} L \cdot X(T_1, T_1, T_n) \cdot \mathbb{1}\left[\tau > \widetilde{T}\right](s_{1,n}(T_1) - s_{\mathrm{K}})^+ & \textit{for the fix payer} \\ L \cdot X(T_1, T_1, T_n) \cdot \mathbb{1}\left[\tau > \widetilde{T}\right](s_{\mathrm{K}} - s_{1,n}(T_1))^+ & \textit{for the fix receiver} \end{cases}$$

is for the fix-payer swaption given by:

$$V_{\mathrm{RS}}(0, \widetilde{T}, T_1, T_n) = L \cdot X(0, T_1, T_n) \cdot (s_{1,n}(0) \cdot A_1 - s_{\mathrm{K}} \cdot A_2),$$

respectively for the fix-receiver swaption

$$V_{\mathrm{RS}}(0, \widetilde{T}, T_1, T_n) = L \cdot X(0, T_1, T_n) \cdot (s_{\mathrm{K}} \cdot A_{-2} - s_{1,n}(0) \cdot A_{-1})$$

where

$$\begin{aligned} A_{\pm 1} &= \int_{-\infty}^{\infty} \exp\left\{au\sigma\sqrt{T_1} - a^2\sigma^2 T_1/2\right\}\Phi\left(\pm\frac{d_1 + au - a^2\sigma\sqrt{T_1}}{\sqrt{1-a^2}}\right) \cdot \\ &\quad \cdot\Phi\left(\frac{bu - \Phi^{-1}(1 - S(\widetilde{T}))}{\sqrt{1-b^2}}\right)\varphi(u)\mathrm{d}u, \\ A_{\pm 2} &= \int_{-\infty}^{\infty} \Phi\left(\pm\frac{d_2 + au}{\sqrt{1-a^2}}\right)\Phi\left(\frac{bu - \Phi^{-1}(1 - S(\widetilde{T}))}{\sqrt{1-b^2}}\right)\varphi(u)\mathrm{d}u, \\ d_1 &= \frac{\log\left(s_{1,n}(0)/s_{\mathrm{K}}\right) + \sigma^2 T_1/2}{\sigma\sqrt{T_1}}, \\ d_2 &= d_1 - \sigma\sqrt{T_1}, \ \textit{and} \\ \varphi(u) &= \frac{1}{\sqrt{2\pi}}e^{-u^2/2}, u \in \mathbb{R}. \end{aligned}$$

The proof of this theorem and following remarks can be in found in Černý and Witzany [2014]. We can use the swaption price to evaluate the CVA of the IRS with n swap payments ($T_1 < \cdots < T_n = T$) starting at the time $t = T_0$ also in the presence of the wrong-way, or right-way, risk. This is achieved by

$$\mathrm{CVA}_{\mathrm{IRS}}(t,T) = \mathrm{LGD} \sum_{i=0}^{n-1} \mathrm{CVA}_{\mathrm{IRS}}(t, T_i, T_{i+1}) \tag{3.3}$$

where $\mathrm{CVA}_{\mathrm{IRS}}(t, T_i, T_{i+1})$ is the expected value of the loss if the counterparty defaults between the times T_i and T_{i+1}. More rigorously, for $i = 0, \ldots, n-1$ we have

$$\begin{aligned}
\mathrm{CVA}_{\mathrm{IRS}}(t, T_i, T_{i+1}) &= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[T_i < \tau \le T_{i+1}\right] D(t,\tau) V(\tau, T)^+\right] \\
&\approx \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[T_i < \tau \le T_{i+1}\right] D(t,T_{i+1}) V(T_{i+1}, T)^+\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[\tau > T_i\right] D(t,T_{i+1}) V(T_{i+1}, T)^+\right. \\
&\quad \left. - \mathbb{1}\left[\tau > T_{i+1}\right] D(t,T_{i+1}) V(T_{i+1}, T)\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[\tau > T_i\right] D(t,T_{i+1}) V(T_{i+1}, T)^+\right] \\
&\quad - \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[\tau > T_{i+1}\right] D(t,T_{i+1}) V(T_{i+1}, T)\right] \\
&= X(0, T_{i+1}, T)\mathbb{E}^{\mathbb{Q}_{i+1,n}}\left[\frac{V_{RS}(T_{i+1}, T_i, T_{i+1}, T_n)}{X(T_{i+1}, T_{i+1}, T)}\right] \\
&\quad - X(0, T_{i+1}, T)\mathbb{E}^{\mathbb{Q}_{i+1,n}}\left[\frac{V_{RS}(T_{i+1}, T_{i+1}, T_{i+1}, T_n)}{X(T_{i+1}, T_{i+1}, T)}\right] \\
&= V_{\mathrm{RS}}(t, T_i, T_{i+1}, T) - V_{\mathrm{RS}}(t, T_{i+1}, T_{i+1}, T).
\end{aligned} \tag{3.4}$$

In other words, the $\mathrm{CVA}_{\mathrm{IRS}}$ at time t with possible default between times T_i and T_{i+1} is approximated by the difference of the risky swaption prices with expiration at T_i and at T_{i+1}, but in both cases with settlement at T_{i+1}, i.e., with payments starting at T_{i+1}. Let us denote this approximation of $\mathrm{CVA}_{\mathrm{IRS}}$ by $\overline{\mathrm{CVA}}_{\mathrm{IRS}}$, i.e.,

$$\begin{aligned}
\overline{\mathrm{CVA}}_{\mathrm{IRS}}(t,T) &= \mathrm{LGD} \sum_{i=0}^{n-1} \overline{\mathrm{CVA}}_{\mathrm{IRS}}(t, T_i, T_{i+1}) \\
&= \mathrm{LGD} \sum_{i=0}^{n-1} \left[V_{\mathrm{RS}}(t, T_i, T_{i+1}, T) - V_{\mathrm{RS}}(t, T_{i+1}, T_{i+1}, T)\right].
\end{aligned} \tag{3.5}$$

The main result of the paper Černý and Witzany [2014] is the theorem which is a semi-analytical formula for IRS CVA calculation involving wrong-way (right-way) risk combined from Theorem1 and (3.5). However, the main weakness of the semi-analytical formula is the Gaussian copula assumption.We realize that the Gaussian distribution is not heavy-tailed and therefore this assumption does not match the situation on the market (see McNeil, Frey, and Embrechts [2005]).

4 Cherubini's Approach

In Cherubini [2013], Fréchet family of copulas describing the perfect dependece between the exposure and the credit quality of the counterparty is used. Consider again a swap with n payments at times $T_1 < \cdots < T_n = T$ with fixed swap rate[2] s_{K}, and τ is the default time of the counterparty. Let $t = T_0$ and let us consider constant LGD. The IRS CVA is calculated using the joint probability in the form

$$\begin{aligned}
&\mathrm{CVA}_{\mathrm{IRS}}(t,T) = \\
&= \ \mathrm{LGD}\sum_{i=0}^{n-1}\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[T_i < \tau \le T_{i+1}\right]D(t,\tau)X(\tau,T_{i+1},T_n)\left(s_{i+1,n}(\tau)-s_{\mathrm{K}}\right)^+\right] \\
&\approx \ \mathrm{LGD}\sum_{i=0}^{n-1}\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}\left[T_i < \tau \le T_{i+1}\right]D(t,T_{i+1})X(T_{i+1},T_{i+1},T_n)\left(s_{i+1,n}(T_{i+1})-s_{\mathrm{K}}\right)^+\right] \\
&= \ \mathrm{LGD}\sum_{i=0}^{n-1}X(t,T_{i+1},T_n)\mathbb{E}_t^{\mathbb{Q}_{i+1,n}}\left[\mathbb{1}\left[T_i < \tau \le T_{i+1}\right]\left(s_{i+1,n}(\tau)-s_{\mathrm{K}}\right)^+\right] \\
&= \ \mathrm{LGD}\sum_{i=0}^{n-1}X(t,T_{i+1},T_n)\int_{s_{\mathrm{K}}}^{\infty}\mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u, T_i < \tau \le T_{i+1}\right].
\end{aligned} \tag{4.1}$$

Now the usage of the copula function is the following: the probability of the above joint event (the swap rate is higher than some fixed treshold and the default occurs between the times T_i and T_{i+1}) is, according to Cherubini [2013], expressed by copula function $\widetilde{\mathbb{C}}$ as

$$\mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u, T_i \le \tau < T_{i+1}\right] = \widetilde{\mathbb{C}}(1-G(u), S(T_i)-S(T_{i+1})) \tag{4.2}$$

where $S(T_{i+1})$ is the risk-neutral survival function of the counterparty at time T_{i+1} and $G(u) = \mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) \le u\right]$ is the absolutely continuous cdf of the swap rate at u. If we put the copula function $\widetilde{\mathbb{C}}$ into (4.1) we obtain

$$\mathrm{CVA}_{\mathrm{IRS}}(t,T) \approx \mathrm{LGD}\sum_{i=0}^{n-1}X(t,T_{i+1},T_n)\int_{s_{\mathrm{K}}}^{\infty}\widetilde{\mathbb{C}}(1-G(u),S(T_i)-S(T_{i+1}))du. \tag{4.3}$$

An analogous expression holds also for the other side of the contract (counterparty is the fix rate receiver)

[2] The fixed swap rate is equal to swap rate at the origination of the IRS contract.

$$\mathrm{CVA_{IRS}}(t,T) \approx \mathrm{LGD}\sum_{i=0}^{n-1} X(t,T_{i+1},T_n)\int_0^{s_\mathrm{K}} \mathbb{C}(G(u),S(T_i)-S(T_{i+1}))du \tag{4.4}$$

where $\mathbb{C}$ is the copula of the joint event of the swap rate being lower than the fixed rate and that the default event is between times T_i and T_{i+1}.

As we mentioned at the beginning of this section, Fréchet copulas are used in the semi-analytical formula, particularly the upper ($\mathbb{C}_U$) and lower ($\mathbb{C}_L$) Fréchet bound, i.e.,

$$\mathbb{C}_U(u,v) = \min\{u,v\} \quad \text{and} \quad \mathbb{C}_L(u,v) = \max\{u+v-1,0\} \tag{4.5}$$

for $0 \le u \le 1$ and $0 \le v \le 1$. In other words, if the counterparty is paying fix rate s_K then the perfect adverse dependence between the exposure and the credit quality (hereinafter perfect WWR) is expressed by the upper Fréchet bound

$$\widetilde{\mathbb{C}}(1-G(u),S(T_i)-S(T_{i+1})) = \min\{1-G(u),S(T_i)-S(T_{i+1})\} \tag{4.6}$$

and IRS CVA with perfect WWR, denoted by $\mathrm{CVA_{IRS}^{WWR}}$, based on $\widetilde{\mathbb{C}}$ is calculated as

$$\begin{aligned}\mathrm{CVA_{IRS}^{WWR}}(t,T) \approx{}& \mathrm{LGD}\sum_{i=0}^{n-1} X(t,T_{i+1},T_n)\max\{k(T_{i+1})-s_\mathrm{K},0\}(S(T_i)-S(T_{i+1}))\\ &+\mathrm{LGD}\sum_{i=0}^{n-1} V_\mathrm{Swaption}(t,T_{i+1},T_n,\max\{s_\mathrm{K},k(T_{i+1})\},1)\end{aligned} \tag{4.7}$$

where $k(T_{i+1}) = \overline{G}^{-1}(S(T_i)-S(T_{i+1}))$, $\overline{G}(u) = 1-G(u)$, and $\overline{G}^{-1}$ is the inverse function of $\overline{G}$. The formula for the other side of the contract is analogous

$$\begin{aligned}\mathrm{CVA_{IRS}^{WWR}}(t,T) \approx{}& \mathrm{LGD}\sum_{i=0}^{n-1} X(t,T_{i+1},T_n)\max\{s_\mathrm{K}-k^*(T_{i+1}),0\}(S(T_i)-S(T_{i+1}))\\ &+\mathrm{LGD}\sum_{i=0}^{n-1} V_\mathrm{Swaption}(t,T_{i+1},T_n,\min\{s_\mathrm{K},k(T_{i+1})\},-1)\end{aligned} \tag{4.8}$$

where $k^*(T_{i+1}) = G^{-1}(S(T_i)-S(T_{i+1}))$. Cherubini [2013] used a local volatility model for the swaption prices in the numerical study, and therefore $k(T_{i+1})$ and $k^*(T_{i+1})$ are such that

$$\begin{aligned}k(T_{i+1}) &= s_{i+1,n}(t)\times\exp\left\{-\frac{\sigma_{i+1}^2}{2}-\Phi^{-1}(S(T_i)-S(T_{i+1}))\,\sigma_{i+1}\sqrt{T_{i+1}-t)}\right\},\\ k^*(T_{i+1}) &= s_{i+1,n}(t)\times\exp\left\{-\frac{\sigma_{i+1}^2}{2}+\Phi^{-1}(S(T_i)-S(T_{i+1}))\,\sigma_{i+1}\sqrt{T_{i+1}-t}\right\}\end{aligned}$$

where $s_{i+1,n}(t)$ is forward swap rate at time t of swap contract starting at time T_{i+1} (tenor is still T_n), σ_{i+1} is volatility corresponding to time T_{i+1} and Φ is the cdf of the standard normal distribution. In order to apply this technique for an arbitrary dependence between exposure and credit quality, Cherubini [2013] uses mixture copula with dependence parameter ρ which reads as

$$\mathrm{CVA}_{\mathrm{IRS}}(t,T) \approx \widetilde{\rho}\,\mathrm{CVA}^{\mathrm{WWR}}_{\mathrm{IRS}}(t,T) + (1-\widetilde{\rho})\mathrm{ICVA}_{\mathrm{IRS}}(t,T). \tag{4.9}$$

where $\mathrm{ICVA}_{\mathrm{IRS}}(t,T)$ is the independent IRS CVA, i.e., IRS CVA without wrong-way risk.

Although the proposed method is very elegant and practitioners friendly, Cherubini [2013] does not offer any calibration of the parameter ρ to real market data.

5 Modified Approach

In further research, we focus on the use of the copula approach for CVA calculation. We were inspired by paper Cherubini [2013] which introduced a IRS CVA semi-analytical formula with the inclusion of wrong-way risk using Fréchet copulas presented in 4.5. As we saw, Cherubini [2013] used the swap rate cdf and the probability of default in a certain time interval as the arguments of the copula. However, in Cherubini [2013], one equality (see Equation (4.2)) which is not completely consistent with Sklar's theorem is used

$$\begin{aligned}&\mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u, T_i \le \tau < T_{i+1}\right] \neq \\ &\qquad \neq \widetilde{\mathbb{C}}(\mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u\right], \mathbb{Q}_{i+1,n}\left[T_i \le \tau < T_{i+1}\right])\end{aligned}$$

which leads to the equation (4.2). The correct representation of the joint probability above in the terms of copulas is the following:

$$\begin{aligned}&\mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u, T_i \le \tau < T_{i+1}\right] = \\ &\quad = \mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u, \tau < T_{i+1}\right] - \mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u, \tau < T_i\right] \\ &\quad = \widetilde{\mathbb{C}}(\mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u\right], \mathbb{Q}_{i+1,n}\left[\tau < T_i\right]) \\ &\qquad - \widetilde{\mathbb{C}}(\mathbb{Q}_{i+1,n}\left[s_{i+1,n}(T_{i+1}) > u\right], \mathbb{Q}_{i+1,n}\left[\tau < T_i\right]).\end{aligned} \tag{5.1}$$

In contrast with Cherubini [2013], we used directly the cdf of the default time as the second argument of the copula so we can use Sklar's theorem. If we combine the results from the sections 2 and 3 we obtain a general copula expression of the expected value of the loss if the counterparty defaults during a certain time period such that

$$\mathrm{CVA}_{\mathrm{IRS}}(0,T_i,T_{i+1}) \approx X(0,T_{i+1},T_n) \int_{s_{\mathrm{K}}}^{\infty} \left(\widetilde{\mathbb{C}}(\overline{G}(s), F(T_{i+1})) - \widetilde{\mathbb{C}}(\overline{G}(s), F(T_i))\right) \mathrm{d}s, \tag{5.2}$$

or

$$\mathrm{CVA}_{\mathrm{IRS}}(0,T_i,T_{i+1}) \approx X(0,T_{i+1},T_n)\int_0^{s_{\mathrm{K}}} \left(\mathbb{C}(G(s),F(T_{i+1})) - \mathbb{C}(G(s),F(T_i))\right)\mathrm{d}s, \tag{5.3}$$

depending on the contract side.

We have recalculated the CVA semi-analytical formula based on upper Frchet copula bounds considering, as well as Cherubini [2013], perfect adverse dependence.

Let us denote by G_{i+1} the cdf of the swap rate $s_{i+1,n}(T_{i+1})$ for $i = 0,\ldots,n-1$, swap rate survival function by $\overline{G}_{i+1}(x) = 1 - G_{i+1}(x)$, and let F be the cdf of the default time τ of the counterparty.

Our results are summarized in the theorem below.

Theorem 2. *Suppose that copulas $\mathbb{C}$, $\widetilde{\mathbb{C}}$ from (5.2), (5.3) are upper Frchet bounds. Furthermore let us suppose that the payoff is postponed to the following swap payment date if the counterparty defaults. Then the $\overline{\mathrm{CVA}}_{\mathrm{IRS}}^{\mathrm{WWR}}$ from the perspective of a fix payer (receiver) is equal to*

$$\begin{aligned}
&\overline{\mathrm{CVA}}_{\mathrm{IRS}}^{\mathrm{WWR}}(t,T) = \\
&= \mathrm{LGD}\sum_{i=0}^{n-1}\Bigg[L\cdot X(0,T_{i+1},T_n)\left[\max\{s_{i+1}-s_{\mathrm{K}},0\}F(T_{i+1}) - \max\{s_i - s_{\mathrm{K}},0\}F(T_i)\right] \\
&+ V_{\mathrm{Swaption}}(0,T_{i+1},T,\max\{s_{i+1},s_{\mathrm{K}}\},1) - V_{\mathrm{Swaption}}(0,T_{i+1},T,\max\{s_i,s_{\mathrm{K}}\},1)\Bigg],
\end{aligned} \tag{5.4}$$

respectively

$$\begin{aligned}
&\overline{\mathrm{CVA}}_{\mathrm{IRS}}^{\mathrm{WWR}}(t,T) = \\
&= \mathrm{LGD}\sum_{i=0}^{n-1}\Bigg[L\cdot X(0,T_{i+1},T_n)\left[\max\{s_{\mathrm{K}}-s^*_{i+1},0\}F(T_{i+1}) - \max\{s_{\mathrm{K}} - s^*_i,0\}F(T_i)\right] \\
&+ V_{\mathrm{Swaption}}(0,T_{i+1},T,\min\{s^*_{i+1},s_{\mathrm{K}}\},-1) - V_{\mathrm{Swaption}}(0,T_{i+1},T,\min\{s^*_i,s_{\mathrm{K}}\},-1)\Bigg]
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
s_{i+1} &= \overline{G}^{-1}_{i+1}(F(T_{i+1})), \quad s_i = \overline{G}^{-1}_{i+1}(F(T_i)), \\
s^*_{i+1} &= G^{-1}_{i+1}(F(T_{i+1})), \quad s^*_i = G^{-1}_{i+1}(F(T_i)).
\end{aligned}$$

Let us recall that the $\overline{\mathrm{CVA}}_{\mathrm{IRS}}$ is an approximation of the $\mathrm{CVA}_{\mathrm{IRS}}$ due to the postponement of the payoff.

In further calculations we make use of the assumption of lognormally distributed swap rates and the assumption of exponentially distributed default time with default intensity h, then

$$\begin{aligned} s_i &= s_{i+1,n}(t) \times \exp\left\{-\frac{\sigma^2}{2} - \Phi^{-1}(1-\exp\{-h(T_i-t)\})\sigma\sqrt{T_{i+1}-t}\right\}, \\ s_{i+1} &= s_{i+1,n}(t) \times \exp\left\{-\frac{\sigma^2}{2} - \Phi^{-1}(1-\exp\{-h(T_{i+1}-t)\})\sigma\sqrt{T_{i+1}-t}\right\} \end{aligned}$$

and

$$\begin{aligned} s_i^* &= s_{i+1,n}(t) \times \exp\left\{-\frac{\sigma^2}{2} + \Phi^{-1}(1-\exp\{-h(T_i-t)\})\sigma\sqrt{T_{i+1}-t}\right\}, \\ s_{i+1}^* &= s_{i+1,n}(t) \times \exp\left\{-\frac{\sigma^2}{2} + \Phi^{-1}(1-\exp\{-h(T_{i+1}-t)\})\sigma\sqrt{T_{i+1}-t}\right\}. \end{aligned}$$

As in Cherubini [2013], we will use a mixture copula given by (4.9) where the dependence between exposure and credit quality is expressed by parameter $\widetilde{\rho}$.

6 Numerical Study

We will price a plain-vanilla at-the-money fix-receiver 10Y IRS (with swap rate 1.15 %) on the EUR market where the fixed leg pays annually a 30E/360 strike rate and the floating leg pays semi-annually LIBOR. The recovery rate is equal to zero, i.e., LGD = 1, and the volatility σ is equal 40 %. Remaining inputs of the model are the zero-bond spot rates which are shown in the Figure 1.

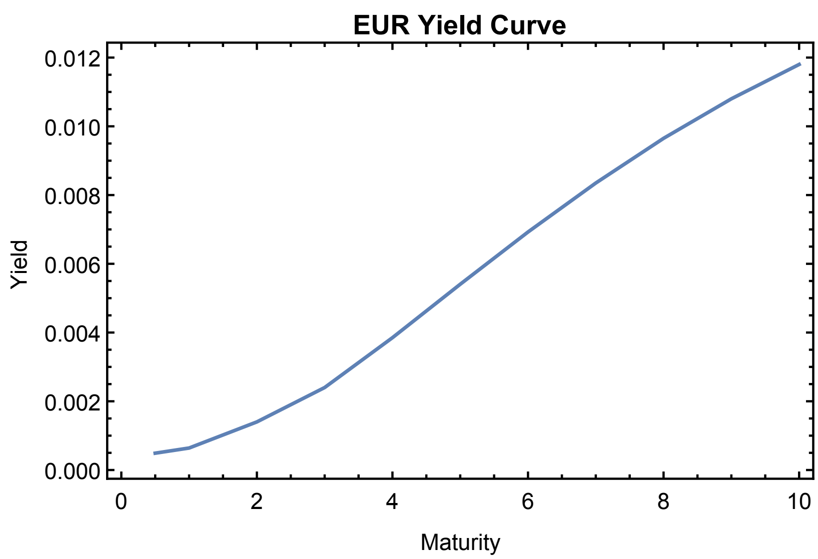

Figure 1: EUR zero-coupon continuously-compounded term structure observed on June,11 2015

We are going to compare the prices of the IRS adjusted for CCR calculated by our semi-analytical approaches, Gaussian copula and modified approach, presented in sections 3 and 5 with different semi-analytical approach proposed by Cherubini [2013] briefly outlined in Section 4.

Table 1 contains the results of all presented approaches calculated without and with the wrong-way risk. The hazard rate h is fixed. The results of our semi-analytical approaches are displayed in the columns CVA_{CW} and CVA_{Mod} where the former correponds to the Gaussian copula approach and the latter to the modified approach. The column CVA_{Cher} contains results of Cherubini's formula results. Note that the correlation used in Gaussian copula approach is the correlation ρ such that

$$\rho = ab = \text{corr}(Y, Z) \tag{6.1}$$

between the levels of the interest rate and the default time, respectively correlation between the random drivers of the interest rate and the default time given by (3.1) and (3.2). The rest of the presented semi-analytical formulas follow similar dependence logic as in the Gaussian copula approach, i.e., between the levels of interest rate and default time ($\rho \approx \widetilde{\rho}$).

corr_{CW}	h	$\mathbf{CVA_{CW}}$	$\mathbf{CVA_{Mod}}$	$\mathbf{CVA_{Cher}}$
$\rho = 0$	3 %	0.240 %	0.240 %	0.240 %
	5 %	0.372 %	0.372 %	0.372 %
	7 %	0.486 %	0.486 %	0.486 %
$\rho = 0.25$	3 %	0.345 %	0.385 %	0.453 %
	5 %	0.501 %	0.553 %	0.692 %
	7 %	0.624 %	0.672 %	0.894 %
$\rho = 0.5$	3 %	0.468 %	0.530 %	0.665 %
	5 %	0.647 %	0.733 %	1.011 %
	7 %	0.776 %	0.858 %	1.301 %
$\rho = 0.75$	3 %	0.621 %	0.674 %	0.877 %
	5 %	0.825 %	0.914 %	1.330 %
	7 %	0.953 %	1.045 %	1.709 %
$\rho = 1$	3 %	0.820 %	0.819 %	1.090 %
	5 %	1.091 %	1.094 %	1.650 %
	7 %	1.220 %	1.231 %	2.117 %

Table 1: IRS prices including CCR with and without wrong-way risk.

The results of CVA_{CW} and CVA_{Mod} with perfect wrong-way risk (i.e., $\rho = 1$) are similar due to the same swap rate distribution. However, the small differences (up to 10 bp) between the results in case of the correlation lower than one are mainly caused by the approximation error of CVAs and also of correlations.

The results of Cherubini's approach are rather illustrative to see how a small inaccuracy in the formula derivation can have a significant impact on the final price of the financial derivative.

7 Conclusion

In this paper we have recalled two methods of semi-analytical calculation of IRS CVA using copula functions, particularly the Gaussian copula approach introduced in Černý and Witzany [2014] and upper Fréchet bound introduced in Cherubini [2013]. We have proposed our semi-analyical IRS CVA formula based on upper Fréchet bound as a modification of Cherubini's approach.

In the numerical study we have compared the results of IRS CVA semi-analytical formulas among each other by pricing 10Y plain-vanilla IRS. In the case of perfect wrong-way risk, the results of Gaussian copula approach and modified approach are almost similar which follows from the properties of these copulas (see McNeil, Frey, and Embrechts [2005]). On the contrary, the results of Cherubini's approach are almost two times higher than the results of other approaches. This is caused by the inconsistency in the application of Sklar's theorem in Cherubini [2013].

We realize that we have artificially set the parameters to observe the behavior of the IRS CVA semi-analytical formulas. In practice, the parameters (mainly the correlation coefficient between the IRS rate and default time) should be calibrated to the market data - IRS and CDS rates. The calibration of the parameters of the Gaussian copula approach can be found in Černý and Witzany [2015]. The calibration of the parameters of the modified approach is a subject of further research.

Appendix A: Proof of Theorem 2

Proof. Let us first consider the fix payer IRS (fix receiver is the risky counterparty). Notice that $\forall i\ s_{i+1} \leq s_i$ and

$$\widetilde{\mathbb{C}}(\overline{G}_{i+1}(s), F(T_{i+1})) - \widetilde{\mathbb{C}}(\overline{G}_{i+1}(s), F(T_i)) = \begin{cases} 0, & \text{for } s_{i+1} < s_i \leq s, \\ F(T_{i+1}) - F(T_i), & \text{for } s \leq s_{i+1}, \\ \overline{G}_{i+1}(s) - F(T_i), & \text{for } s_{i+1} < s < s_i. \end{cases}$$

Hence,

$$\int_{s_{\mathrm{K}}}^{\infty} \widetilde{\mathbb{C}}(\overline{G}_{i+1}(s), F(T_{i+1})) - \widetilde{\mathbb{C}}(\overline{G}_{i+1}(s), F(T_i)) =$$

$$= \max\{s_{i+1} - s_{\mathrm{K}}, 0\}(F(T_{i+1}) - F(T_i)) + \int_{\max\{s_{i+1}, s_{\mathrm{K}}\}}^{\max\{s_i, s_{\mathrm{K}}\}} (\overline{G}_{i+1}(s) - F(T_i))\mathrm{d}s \tag{7.1}$$

$$= \max\{s_{i+1} - s_{\mathrm{K}}, 0\}F(T_{i+1}) - \max\{s_i - s_{\mathrm{K}}, 0\}F(T_i) + \int_{\max\{s_{i+1}, s_{\mathrm{K}}\}}^{\max\{s_i, s_{\mathrm{K}}\}} \overline{G}_{i+1}(s))\mathrm{d}s$$

and the integral on the right-hand side is in fact the expected value of the swaption payoff with a modified strike rate

$$\begin{aligned} \int_{\max\{s_{i+1}, s_{\mathrm{K}}\}}^{\max\{s_i, s_{\mathrm{K}}\}} \overline{G}_{i+1}(s)\mathrm{d}s &= \int_{\max\{s_{i+1}, s_{\mathrm{K}}\}}^{\infty} \overline{G}_{i+1}(s)\mathrm{d}s - \int_{\max\{s_i, s_{\mathrm{K}}\}}^{\infty} \overline{G}_{i+1}(s)\mathrm{d}s \\ &= \mathbb{E}_0^{\mathbb{Q}}\left[(s_{i+1,n}(T_{i+1}) - \max\{s_{i+1}, s_{\mathrm{K}}\})^+\right] \\ &\quad -\mathbb{E}_0^{\mathbb{Q}}\left[(s_{i+1,n}(T_{i+1}) - \max\{s_i, s_{\mathrm{K}}\})^+\right] \end{aligned}$$

Finally, multiplying equation (7.1) by $X(0, T_{i+1}, T_n)$ and L we get that

$$\begin{aligned} &\overline{\mathrm{CVA}}_{\mathrm{IRS}}^{\mathrm{WWR}}(0, T_i, T_{i+1}) = \\ &\quad = L \cdot X(0, T_{i+1}, T_n)\left[\max\{s_{i+1} - s_{\mathrm{K}}, 0\}F(T_{i+1}) - \max\{s_i - s_{\mathrm{K}}, 0\}F(T_i)\right] \\ &\qquad + V_{\mathrm{Swaption}}(0, T_{i+1}, T, \max\{s_{i+1}, s_{\mathrm{K}}\}, 1) - V_{\mathrm{Swaption}}(0, T_{i+1}, T, \max\{s_i, s_{\mathrm{K}}\}, 1). \end{aligned}$$

The calculation of $\overline{\mathrm{CVA}}_{\mathrm{IRS}}^{\mathrm{WWR}}(t, T)$ is now straight-forward from the first equation in (3.5).

The proof of the second formula, with a few exceptions, is very similar to the first one. Notice that $\forall i\ s_i^* \leq s_{i+1}^*$ and

$$\mathbb{C}(G_{i+1}(s), F(T_{i+1})) - \mathbb{C}(G_{i+1}(s), F(T_i)) = \begin{cases} 0 & \text{for } s \leq s_i^* < s_{i+1}^*, \\ F(T_{i+1}) - F(T_i) & \text{for } s_i^* < s_{i+1}^* \leq s, \\ G_{i+1}(s) - F(T_i) & \text{for } s_i^* < s < s_{i+1}^*. \end{cases}$$

Hence

$$\int_0^{s_\mathrm{K}} \mathbb{C}(G_{i+1}(s), F(T_{i+1})) - \mathbb{C}(G_{i+1}(s), F(T_i)) =$$

$$= \max\{s_\mathrm{K} - s^*_{i+1}, 0\}(F(T_{i+1}) - F(T_i)) + \int_{\min\{s^*_i, s_\mathrm{K}\}}^{\min\{s^*_{i+1}, s_\mathrm{K}\}} (G_{i+1}(s) - F(T_i))\mathrm{d}s$$

$$= \max\{s_\mathrm{K} - s^*_{i+1}, 0\}F(T_{i+1}) - \max\{s_\mathrm{K} - s^*_i, 0\}F(T_i) + \int_{\min\{s^*_i, s_\mathrm{K}\}}^{\min\{s^*_{i+1}, s_\mathrm{K}\}} G_{i+1}(s)\mathrm{d}s, \tag{7.2}$$

because

$$\begin{aligned} \min\{s^*_{i+1}, s_\mathrm{K}\} - \min\{s^*_i, s_\mathrm{K}\} &= \min\{s^*_{i+1} - s_\mathrm{K}, 0\} - \min\{s^*_i - s_\mathrm{K}, 0\} \\ &= -\max\{s_\mathrm{K} - s^*_{i+1}, 0\} + \max\{s_\mathrm{K} - s^*_i, 0\}. \end{aligned}$$

Then, having computed the last integral, we get expected values of swaption payoff with modified strike rates different from the previous case as

$$\begin{aligned} \int_{\min\{s^*_i, s_\mathrm{K}\}}^{\min\{s^*_{i+1}, s_\mathrm{K}\}} G_{i+1}(s)\mathrm{d}s &= \int_0^{\min\{s^*_{i+1}, s_\mathrm{K}\}} G_{i+1}(s)\mathrm{d}s - \int_0^{\min\{s^*_i, s_\mathrm{K}\}} G_{i+1}(s)\mathrm{d}s \\ &= \mathbb{E}_0^{\mathbb{Q}}\left[\left(\min\{s^*_{i+1}, s_\mathrm{K}\} - s_{i+1,n}(T_{i+1})\right)^+\right] \\ &\quad -\mathbb{E}_0^{\mathbb{Q}}\left[\left(\min\{s^*_i, s_\mathrm{K}\} - s_{i+1,n}(T_{i+1})\right)^+\right]. \end{aligned}$$

Again, multiplying equation (7.2) by $X(0, T_{i+1}, T_n)$ and L yields

$$\begin{aligned} &\overline{\mathrm{CVA}}^{\mathrm{WWR}}_{\mathrm{IRS}}(0, T_i, T_{i+1}) = \\ &= L \cdot X(0, T_{i+1}, T_n)\left[\max\{s^*_{i+1} - s_\mathrm{K}, 0\}F(T_{i+1}) - \max\{s^*_i - s_\mathrm{K}, 0\}F(T_i)\right] \\ &\quad + V_{\mathrm{Swaption}}(0, T_{i+1}, T, \min\{s^*_{i+1}, s_\mathrm{K}\}, -1) - V_{\mathrm{Swaption}}(0, T_{i+1}, T, \min\{s^*_i, s_\mathrm{K}\}, -1). \end{aligned}$$

which completes the proof.

□

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